

# Abstract Interpretation with Infinitesimals <sup>★</sup>

## Towards Scalability in *Nonstandard Static Analysis*

Kengo Kido<sup>1,2</sup>, Swarat Chaudhuri<sup>3</sup>, and Ichiro Hasuo<sup>1</sup>

<sup>1</sup> University of Tokyo, Japan

<sup>2</sup> JSPS Research Fellow

<sup>3</sup> Rice University, USA

**Abstract.** We extend abstract interpretation for the purpose of verifying hybrid systems. Abstraction has been playing an important role in many verification methodologies for hybrid systems, but some special care is needed for abstraction of continuous dynamics defined by ODEs. We apply Cousot and Cousot’s framework of abstract interpretation to hybrid systems, almost *as it is*, by regarding continuous dynamics as an infinite iteration of *infinitesimal* discrete jumps. This extension follows the recent line of work by Suenaga, Hasuo and Sekine, where deductive verification is extended for hybrid systems by 1) introducing a constant  $\text{dt}$  for an infinitesimal value; and 2) employing Robinson’s *nonstandard analysis* (NSA) to define mathematically rigorous semantics. Our theoretical results include soundness and termination via *uniform* widening operators; and our prototype implementation successfully verifies some benchmark examples.

## 1 Introduction

*Hybrid systems* exhibit both discrete *jump* and continuous *flow* dynamics. Quality assurance of such systems are of paramount importance due to the current ubiquity of *cyber-physical systems* (CPS) like cars, airplanes, and many others. For the formal verification approach to hybrid systems, the challenges are: 1) to incorporate flow-dynamics; and 2) to do so at the lowest possible cost, so that the existing discrete framework smoothly transfers to hybrid situations. A large body of existing work uses *differential equations* explicitly in the syntax; see the discussion of related work below.

In [33], instead, an alternative approach of *nonstandard static analysis*—combining *static analysis* and *nonstandard analysis*—is proposed. Its basic idea is to introduce a constant  $\text{dt}$  for an *infinitesimal* (i.e. infinitely small) value, and *turn flow into jump*. With  $\text{dt}$ , the continuous operation of integration can be represented by a while-loop, to which existing discrete techniques such as Hoare-style program logics readily apply. For a rigorous mathematical development they employ *nonstandard analysis* (NSA) beautifully formalized by Robinson [32].

Concretely, in [33] they took the common combination of a WHILE-language and a Hoare logic (e.g. in the textbook [35]); and added a constant  $\text{dt}$  to obtain a modeling and verification framework for hybrid systems. Its components are called  $\text{WHILE}^{\text{dt}}$

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and  $\text{HOARE}^{\text{dt}}$ . The soundness of  $\text{HOARE}^{\text{dt}}$  is proved against denotational semantics defined in the language of NSA. Subsequently in the *nonstandard static analysis* program: in [22] they presented a prototype automatic theorem prover for  $\text{HOARE}^{\text{dt}}$ ; and in [34] they applied the same idea to stream processing systems, realizing a verification framework for *signal processing* as in Simulink.

Underlying these technical developments is the idea of so-called *sectionwise execution*. Although this paper does not rely explicitly on it, it is still useful for laying out the “operational” intuition of nonstandard static analysis. See the following example.

**Example 1.1.** Let  $c_{\text{elapse}}$  be the program on the right. The value of  $\text{dt}$  is infinitesimal; therefore the while loop will not terminate within finitely many steps. Nevertheless it is somehow intuitive to expect that after an “execution” of this program, the value of  $t$  should be infinitesimally close to 1 and larger than it.

One possible way of thinking is to imagine *sectionwise execution*. For each natural number  $i$  we consider the  $i$ -th section of the program  $c_{\text{elapse}}$ , denoted by  $c_{\text{elapse}}|_i$  and shown on the right. Concretely,  $c_{\text{elapse}}|_i$  is obtained by replacing the infinitesimal  $\text{dt}$  in  $c_{\text{elapse}}$  with  $\frac{1}{i+1}$ . Informally  $c_{\text{elapse}}|_i$  is the “ $i$ -th approximation” of the original  $c_{\text{elapse}}$ .

A section  $c_{\text{elapse}}|_i$  does terminate within finite steps and yields  $1 + \frac{1}{i+1}$  as the value of  $t$ . Now we collect the outcomes of sectionwise executions and obtain a sequence

$$(1 + 1, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \dots, 1 + \frac{1}{i}, \dots) \quad (1)$$

which is thought of as a progressive approximation of the actual outcome of the original program  $c_{\text{elapse}}$ . Indeed, in the language of NSA, the sequence (1) represents a *hyperreal number*  $r$  that is infinitesimally close to 1.

We note that a program in  $\text{WHILE}^{\text{dt}}$  is *not* intended to be executed: the program  $c_{\text{elapse}}$  does not terminate. It is however an advantage of *static* approaches to verification and analysis, that programs need not be executed to prove their correctness. Instead well-defined mathematical semantics suffices. This is what we do here as well as in [22, 33, 34], with the denotational semantics of  $\text{WHILE}^{\text{dt}}$  exemplified in Example 1.1.

**Our Contribution** In the previous work [22, 33, 34] *invariant discovery* has been a big obstacle in scalability of the proposed verification techniques—as is usual in deductive verification. The current work, as a first step towards scalability of the approach, extends *abstract interpretation* [10] with infinitesimals. The abstract interpretation methodology is known for its ample applicability (it is employed in model checking as well as in many deductive verification frameworks) and scalability (the static analyzer Astrée [12] has been successfully used e.g. for Airbus’s flight control system).

Our theoretical contribution includes: the theory of *nonstandard abstract interpretation* where (standard) abstract domains are “\*-transformed,” in a rigorous NSA sense, to the abstract domains for hyperreals; their soundness in over-approximating semantics of  $\text{WHILE}^{\text{dt}}$  programs and hybrid system modeling by them; and introduction of the notion of *uniform* widening operators. With the latter, inductive approximation is guaranteed to terminate within finitely many steps—even after extension to the nonstandard setting. We show that many known widening operators, if not all, are indeed uniform.

Although we focus on the domain of convex polyhedra in this paper, it is also possible to extend other abstract domains like ellipsoids [14] in the same way.

These theoretical results form a basis of our prototype implementation,<sup>4</sup> that successfully analyzes: *water-level monitor*, a common example of piecewise-linear hybrid dynamics; and also *thermostat* that is beyond piecewise-linear. The prototype deals with the constant  $\text{dt}$  as a truly infinitesimal number using computer algebra system.

**Related Work** There has been a lot of research work for verification of hybrid systems and it has led to quite a few system verification tools, including HyTech [25], PHAVer [16], SpaceEx [17], HySAT/iSAT [15], Flow\* [5] and KeYmaera [31]. All these rely on ODEs (or the explicit solutions of them) for expressing continuous dynamics, much like *hybrid automata* [1] do.

Our nonstandard static analysis approach is completely different from those in the following point: we do not use ODEs at all, and model hybrid systems as an imperative program with an infinitesimal constant. It enables us to apply static methodologies for discrete systems as they are. For example, in HyTech and PHAVer, convex polyhedra is used to over-approximate the reachable sets. They need, however, some special techniques such as linear phase-portrait [24], to reduce the dynamics into piecewise linear one. Our framework does not need such and usual abstract interpretation works as it is.

There are many other works we rely on, such as those on abstract interpretation, nonstandard analysis, etc. These are discussed later when they become relevant.

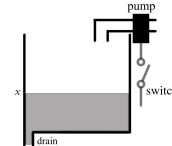
**Organization** In §2 we start with the water-level monitor example and present how our nonstandard abstract interpretation framework works. Then we go on to its theoretical foundations. In §3 we review preliminaries on: abstract interpretation; nonstandard analysis; and the modeling language WHILE<sup>dt</sup> from [33]. In §4 we extend the theory of abstract interpretation with infinitesimals and build the theory of nonstandard abstract interpretation. Its theorems include soundness of approximation, and termination guaranteed by (the  $*$ -transform of) a *uniform* widening operator. In §5 we present our prototype implementation and the experiment results with it.

Most proofs are deferred to Appendix C.

## 2 Leading Example: Verification of Water-Level Monitor

We shall start with an example of verification and let it exemplify how our framework—that extends abstract interpretation with infinitesimals, and handles continuous as well as discrete dynamics—works. We use the well-known example of the water-level monitor [1]. In the current section, in particular, we will first revisit how the usual abstract interpretation workflow (without extension) would work, using a discretized variant of the problem. Our emphasis is on the fact that our extended framework works just in the same manner: without any explicit ODEs or any additional theoretical infrastructure for ODEs; but only adding a constant  $\text{dt}$ .

The concrete problem is as follows. See the figure on the right. A water tank has a constant drain (2 cm per second). When the water level  $x$  gets lower than 5 cm the switch is turned on, which eventually



<sup>4</sup> The prototype is available on-line: <http://www-mmm.is.s.u-tokyo.ac.jp/~kkido/>

makes the pump work but only after a time lag of two seconds. While the pump is working, the water level  $x$  rises by 1 cm per second. Once  $x$  reaches 10 cm the switch is turned off, which will shut down the pump but again after a time lag of two seconds. Our goal is the *reachability analysis* of this hybrid dynamics, that is, to see the water level  $x$  remains in a certain “safe” range (we will see that the range is  $1 \leq x \leq 12$ ).

## 2.1 Analysis by (Standard) Abstract Interpretation, as a Precursor

Let us first revisit the usual workflow in reachability analyses by abstract interpretation. We will use the *discretized* model of the water-level monitor in Fig. 1, where each iteration of its unique loop amounts to the lapse of  $dt' = 0.2$  seconds. The model in Fig. 1 is an imperative program with while loops, a typical subject of analyses by abstract interpretation.

More specifically:  $x$  is the water level,  $l$  is the counter for the time lag,  $p$  stands for the state of the pump ( $p = 0$  if the pump is off, and  $p = 1$  if on) and  $s$  is for “signals,” meaning  $s = 1$  if the pump has not yet responded to a signal from the switch (such as, when the switch is on but the pump is not on yet).

The first step in the usual abstract interpretation workflow is to fix *concrete* and *abstract domains*. Here in §2.1 we will use the followings.

- **The concrete domain:**  $(\mathcal{P}(\mathbb{R}^2))^4$ . We have two numerical variables  $l, x$  and two Boolean ones  $p, s$  in Fig. 1, therefore a canonical concrete domain would be  $\mathcal{P}(\mathbb{B}^2 \times \mathbb{R}^2)$ . We have the powerset operation  $\mathcal{P}$  in it since we are now interested in the *reachable* set of memory states. However, for a better fit with our abstract domain (namely convex polyhedra), we shall use the set  $(\mathcal{P}(\mathbb{R}^2))^4$  that is isomorphic to the above set  $\mathcal{P}(\mathbb{B}^2 \times \mathbb{R}^2)$ .
- **The abstract domain:**  $(\mathbb{CP}_2)^4$ . We use the domain of *convex polyhedra* [13], one of the most commonly-used abstract domains. Recall that a convex polyhedron is a subset of a Euclidean space characterized by a finite conjunction of linear inequalities. Specifically, we let  $\mathbb{CP}_2$ , the set of 2-dimensional convex polyhedra, approximate the set  $\mathcal{P}(\mathbb{R}^2)$ . Therefore, as an abstract domain for the program in Fig. 1, we take  $(\mathbb{CP}_2)^4$  (that approximates  $(\mathcal{P}(\mathbb{R}^2))^4$ ).

The next step in the workflow is to *over-approximate* the set of memory states that are reachable by the program in Fig. 1—this is a subset of the concrete domain  $(\mathcal{P}(\mathbb{R}^2))^4$ —using the abstract domain  $(\mathbb{CP}_2)^4$ . Since the desired set can be thought of as a least fixed point, this over-approximation procedure involves: 1) *abstract execution* of the program in  $(\mathbb{CP}_2)^4$  (that is straightforward, see e.g. [13]); and 2) acceleration of least fixed-point computation in  $(\mathbb{CP}_2)^4$  via suitable use of a *widening operator*. For

```
(*Water-Level Monitor*)
l := 0; x := 1; p := 1; s := 0;
dt' := 0.2;
while true do {
  if p = 1 then x := x + dt'
  else x := x - 2 * dt';
  if (x <= 5 && p = 0) then s := 1
  else if (x >= 10 && p = 1)
    then s := 1
    else s := 0
};
if s = 1 then l := l + dt'
else skip;
if s = 1 && l >= 2
then {p := 1 - p; s := 0; l := 0}
else skip
}
```

**Fig. 1.** Discretized water-level monitor

convex polyhedra several widening operators are well-known. We shall use here  $\nabla_M$ , so-called the *widening up to  $M$*  operator from [20, 21]. One big reason for this choice is the *uniformity* of the operator (a notion we introduce later in §4.3), among others. The set  $M$  of linear constraints is a parameter for this widening operator; we fix it as usual, collecting the linear constraints that occur in the program in question. That is,  $M = \{x \leq 5, x \geq 5, x \leq 10, x \geq 10, l \leq 2, l \geq 2\}$ .

This over-approximation procedure is depicted in the *iteration sequence* in Fig. 3. Let us look at some of its details. The graph 0 represents the initial memory state (before the first iteration), where the pump is on and the water level  $x$  is precisely 1. After one iteration the water level will be incremented by  $1 \times dt' = 0.2$  cm; as usual in abstract interpretation, however, at this moment we invoke the widening operator  $\nabla_M$ , and the next “abstract reachable set” is  $x \in [1, 5]$  instead of  $x \in [1, 1.2]$ . Here the upper bound 5 comes from the constraint  $x \leq 5$  that is in the parameter  $M$  of the widening operator  $\nabla_M$ . This results in the graph 1 in Fig. 3.

In the iteration sequence (Fig. 3) the four polyhedra (in four different colors) gradually grow: in the graph 2 the water level  $x$  can be 10 cm so in the graph 3 appears a green polyhedron (meaning that a signal is sent from the switch to the pump); after the graphs 3 and 9 we *delay* widening, a heuristic commonly employed in abstract interpretation [9]. In the end, in the graph 12 we have a prefixed point (meaning that the polyhedra do not grow any further). There we can see, from the range of  $x$  spanned by the polyhedra, that the water level never reaches beyond  $0.6 \leq x \leq 12.2$ .

## 2.2 Analysis by Nonstandard Abstract Interpretation

In the above “standard” scenario, we approximated the dynamics of the water level by discretizing the continuous notion of time ( $dt' = 0.2$ ). While this made the usual abstract interpretation workflow go around, there is a price to pay—the analysis result is not *precise*. Specifically, the reachable region thus over-approximated is  $0.6 \leq x \leq 12.2$ , while the real reachable region is  $1 \leq x \leq 12$ .<sup>5</sup>

Obviously we can “tighten up” the analysis by making the value  $dt'$  smaller. Even better, we can leave the expression  $dt'$  in Fig. 1 as a variable, and imagine the “limit” of analysis results when the value of  $dt'$  tends to 0. However here is a question: what is that “limit,” in mathematically rigorous terms? Taking  $dt' = 0$  obviously does not work: do so in Fig. 1 and we have no dynamics whatsoever. The value of  $dt'$  must be strictly positive.

Our contribution is an extension of abstract interpretation that answers the last question. In our framework, the same (hybrid) dynamics of the water-level monitor is modeled by a program in Fig. 2. Here the expression  $dt$  is a new constant that stands

```
(*Water-Level Monitor*)
l := 0; x := 1; p := 1; s := 0;
while true do {
  if p = 1 then x := x + dt
  else x := x - 2 * dt;
  if (x <= 5 && p = 0) then s := 1
  else {if (x >= 10 && p = 1)
        then s := 1
        else s := 0
      };
  if s = 1 then l := 1 + dt
  else skip;
  if s = 1 && l >= 2
  then {p := 1 - p; s := 0; l := 0}
  else skip
}
```

**Fig. 2.** Water-level monitor in WHILE<sup>dt</sup>

<sup>5</sup> There are also examples in which discretization even leads to *unsound* analysis results.

for a *positive* and *infinitesimal* (i.e. infinitely small) value. Therefore the modeling is not an approximation by discretization; it is an *exact* modeling.

It is important to notice that the program in Fig. 2 is the same as the one in Fig. 1, except that now  $dt$  is some strange constant, while  $dt'$  in Fig. 1 stood for a real number (namely 0.2). This difference, however, does not prevent us from applying the *static*, *symbolic* and *syntax-based* analysis by abstract interpretation. We can follow exactly the same path as in §2.1—taking the abstract domain of convex polyhedra, executing the program in Fig. 2 on it, applying the widening operator  $\nabla_M$ , and forming an iteration sequence much like in Fig. 3—and this leads to the analysis result  $1 - 2dt \leq x \leq 12 + dt$ . Since  $dt$  is an infinitesimal number, the last result is practically as good as  $1 \leq x \leq 12$ . We have a prototype implementation that automates this analysis (§5).

What remains to be answered is the legitimacy of this extended abstract interpretation framework. Is the outcome  $1 - 2dt \leq x \leq 12 + dt$  *sound*, in the sense that it indeed over-approximates the true reachable set? Even before that, what do we mean by the “true reachable set” of the program in Fig. 2, with an exotic infinitesimal constant like  $dt$ ? Moreover, are iteration sequences via the widening operator  $\nabla_M$  guaranteed to terminate within finitely many steps, as is the case in the standard framework [20, 21]?

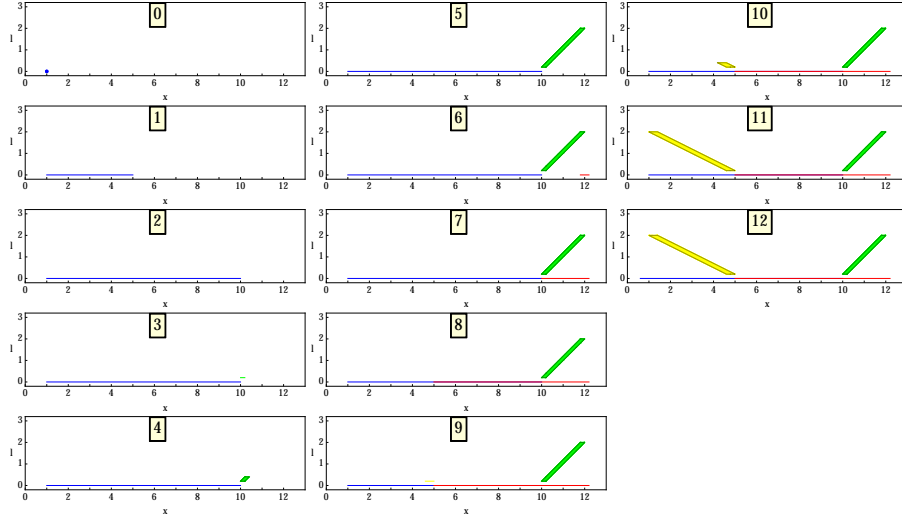
The rest of the paper is mostly devoted to (answering positively to) the last questions. In it we use Robinson’s *nonstandard analysis* (NSA) [32] and give infinitesimal numbers—clearly such do not exist in the set of (standard) real numbers—a status as first-class citizens. The program in Fig. 2 is in fact in the programming (or rather *modeling*) language  $\text{WHILE}^{dt}$  from [22, 33]; and its semantics can be understood in the line of Example 1.1. It turns out that the theory of NSA—in particular its celebrated result of the *transfer principle*—allows us to “transfer” meta results from the standard abstract interpretation to our extension. That is, what is true in the world of standard reals (soundness, termination, etc.) is also true in that of *hyperreals*.

### 3 Preliminaries

In §4 we will present our *soundness* and *termination* results as a “metatheory” that justifies the workflow described in §2.2; in this section we recall some preliminaries that are needed for those theoretical developments. First, the general theory of abstract interpretation is introduced in §3.1 and the specific domain of convex polyhedra is presented in §3.2. Next, some basic notions in nonstandard analysis are explained in §3.3. Finally, in §3.4, the modeling language  $\text{WHILE}^{dt}$  from [33] and its (denotational) collecting semantics based on nonstandard analysis are presented.

#### 3.1 Abstract Interpretation

*Abstract interpretation* [13] is a well-established technique in static analysis. We make a brief review of its basic theory; it is mostly for the purpose of fixing notations. The goal of abstract interpretation is to over-approximate a *concrete semantics* defined on an *concrete domain* by an *abstract semantics* on an *abstract domain*. We assume that the concrete semantics is defined as a least fixed point on the concrete domain. The following proposition guarantee the over-approximation of the least fixed point in the



**Fig. 3.** An iteration sequence for the water-level monitor example.

To save space, here we depict an element of  $(\mathbb{CP}_2)^4$ —i.e. a quadruple of convex polyhedra—on the same plane  $\mathbb{R}^2$ . The four convex polyhedra come in different colors: those in blue, green, red and yellow correspond to the values  $(p, s) = (1, 0), (1, 1), (0, 0)$  and  $(0, 1)$  of the Boolean variables, respectively.

concrete domain by a prefixed point in the abstract domain. In the proposition, the order  $\sqsubseteq$  on the domain  $L$  is extended to the order on  $L \rightarrow L$  pointwisely. And the *least fixed point relative to  $\perp$* , denoted by  $\text{lfp}_{\perp} F$ , is the least among the fixed points of  $F$  above  $\perp$ ; by the cpo structure of  $L$  and the continuity of  $F$ , it is given by  $\bigsqcup_{n \in \mathbb{N}} F^n \perp$ . Note that we are using the concretization-based framework described in [11].

**Proposition 3.1.** *Let  $(L, \sqsubseteq)$  be a cpo;  $F : L \rightarrow L$  be a continuous function; and  $\perp \in L$  be such that  $\perp \sqsubseteq F(\perp)$ . Let  $(\bar{L}, \sqsubseteq)$  be a preorder;  $\gamma : \bar{L} \rightarrow L$  be a function (it is called concretization) such that  $\bar{a} \sqsubseteq \bar{b} \Rightarrow \gamma(\bar{a}) \sqsubseteq \gamma(\bar{b})$  for all  $\bar{a}, \bar{b} \in \bar{L}$ ; and  $\bar{F} : \bar{L} \rightarrow \bar{L}$  be a monotone function such that  $F \circ \gamma \sqsubseteq \gamma \circ \bar{F}$ . Assume further that  $\bar{x} \in \bar{L}$  is a prefixed point of  $\bar{F}$  (i.e.  $\bar{F}(\bar{x}) \sqsubseteq \bar{x}$ ) such that  $\perp \sqsubseteq \gamma(\bar{x})$ .*

*Then  $\bar{x}$  over-approximates  $\text{lfp}_{\perp} F$ , that is,  $\text{lfp}_{\perp} F \sqsubseteq \gamma(\bar{x})$ .*  $\square$

In §2.1 where we analyzed the discretized water-level monitor, the set  $\mathcal{P}(\mathbb{R}^n)$  of subsets of memory states is used as a concrete domain  $L$ ; and the *domain of convex polyhedra* is used as an abstract domain  $\bar{L}$ . The interpretations  $F$  and  $\bar{F}$  on each domains are defined in a standard manner. Towards the goal of obtaining  $\bar{x}$  in Prop. 3.1, (i.e. finding a prefixed point in the abstract domain), the following notion of *widening* is used (often together with *narrowing* that we will not be using). Note that in the following definition and proposition, the domain  $(L, \sqsubseteq)$  is the abstract domain, corresponding to  $(\bar{L}, \sqsubseteq)$  in Prop. 3.1.

**Definition 3.2** (widening operator). Let  $(L, \sqsubseteq)$  be a preorder. A function  $\nabla : L \times L \rightarrow L$  is said to be a *widening operator* if the following two conditions hold.



- (Covering) For any  $x, y \in L$ ,  $x \sqsubseteq x \nabla y$  and  $y \sqsubseteq x \nabla y$ .
- (Termination) For any ascending chain  $\langle x_i \rangle \in L^\mathbb{N}$ , the chain  $\langle y_i \rangle \in L^\mathbb{N}$  defined by  $y_0 = x_0$  and  $y_{i+1} = y_i \nabla x_{i+1}$  for each  $i \in \mathbb{N}$  is ultimately stationary.

A widening operator on a fixed abstract domain  $\bar{L}$  is not at all unique. In this paper we will discuss three widening operators previously introduced for  $\mathbb{CP}_n$ .

The use of widening is as in the following proposition: the covering condition ensures that the outcome is a prefixed point; and the procedure terminates thanks to the termination condition.

**Proposition 3.3** (convergence of iteration sequences). *Let  $(L, \sqsubseteq)$  be a preorder;  $F : L \rightarrow L$  be a monotone function;  $\perp \in L$  be such that  $\perp \sqsubseteq F(\perp)$ ;  $\nabla : L \times L \rightarrow L$  be a widening operator; and  $\langle X_i \rangle_{i \in \mathbb{N}} \in L^\mathbb{N}$  be the infinite sequence defined by*

$$X_0 = \perp ; \quad \text{and, for each } i \in \mathbb{N}, \quad X_{i+1} = \begin{cases} X_i & (\text{if } F(X_i) \sqsubseteq X_i) \\ X_i \nabla F(X_i) & (\text{otherwise}) \end{cases}$$

*Then the sequence  $\langle X_i \rangle_{i \in \mathbb{N}}$  is increasing and ultimately stationary; moreover its limit  $\bigsqcup_{i \in \mathbb{N}} X_i$  is a prefixed point of  $F$  such that  $\perp \sqsubseteq \bigsqcup_{i \in \mathbb{N}} X_i$ .  $\square$*

### 3.2 The Domain of Convex Polyhedra

The domain of convex polyhedra, introduced in [13], is one of the most commonly used relational numerical abstract domains.

**Definition 3.4** (domain of convex polyhedra  $\mathbb{CP}_n$ ). An  $n$ -dimensional *convex polyhedron* is the intersection of finitely many (closed) affine half-spaces. We denote the set of convex polyhedra in  $\mathbb{R}^n$  by  $\mathbb{CP}_n$ . Its preorder  $\sqsubseteq$  is given by the inclusion order (actually it is a partial order). The concretization function  $\gamma_{\mathbb{CP}_n} : \mathbb{CP}_n \rightarrow \mathcal{P}(\mathbb{R}^n)$  is defined in an obvious manner.

We will be studying three widening operators on  $\mathbb{CP}_n$ . They are namely: the *standard widening operator*  $\nabla_S$  [19];<sup>6</sup> the *widening operator*  $\nabla_M$  up to  $M$  [20, 21]; and the *precise widening operator*  $\nabla_N$  [3]. We briefly describe the former two; the definition of the last is omitted for the lack of space. In the following definitions, the function  $\text{con}$  maps a set of linear constraints (called a *constraint system*) to the convex polyhedron induced by the conjunction of those linear constraints.

**Definition 3.5** (standard widening  $\nabla_S$ ). Let  $P_1, P_2 \in \mathbb{CP}_n$ ; and  $C_1$  and  $C_2$  be constraints system that induce  $P_1$  and  $P_2$ , respectively. The *standard widening operator*  $\nabla_S : \mathbb{CP}_n \times \mathbb{CP}_n \rightarrow \mathbb{CP}_n$  is defined by

$$P_1 \nabla_S P_2 := \begin{cases} P_2 & \text{if } P_1 = \emptyset \\ \text{con} \left( \begin{array}{l} \{\varphi \in C_1 \mid C_2 \text{ implies } \varphi, \text{ i.e. } \varphi \text{ is everywhere true in } P_2\} \\ \cup \{\psi \in C_2 \mid \exists \varphi \in C_1. P_1 = \text{con}(C_1[\psi/\varphi])\} \end{array} \right) & \text{otherwise.} \end{cases}$$

<sup>6</sup> The name “standard” is confusing with the distinction between *standard* and *nonstandard* entities in NSA. The use of “standard” in the former sense is scarce in this paper.



Intuitively  $P_1 \nabla_S P_2$  is represented by the set of those linear constraints of  $P_1$  which are satisfied by every point of  $P_2$ .

The following second widening operator  $\nabla_M$  refines  $\nabla_S$ . This is what we use in our implementation. Here  $M$  is a parameter.

**Definition 3.6** (widening up to  $M$ ,  $\nabla_M$ ). Let  $P_1, P_2 \in \mathbb{CP}_n$ , and  $M$  be a (given) finite set of linear inequalities. The *widening operator up to  $M$*  is defined by

$$P_1 \nabla_M P_2 := (P_1 \nabla_S P_2) \cap \text{con}(\{\varphi \in M \mid P_i \subseteq \text{con}(\{\varphi\}) \text{ for } i = 1, 2\}) .$$

The parameter  $M$  is usually taken to be the set of linear inequalities that occur in the program under analysis.

### 3.3 Nonstandard Analysis

Here we list a minimal set of necessary definitions and results in nonstandard analysis (NSA) [32]. Some further details can be found in Appendix A; fully-fledged and accessible expositions of NSA are found e.g. in [18, 26].

The following notions will play important roles.

- *Hyperreals* that extends reals by infinitesimals, infinities, etc.;
- The *transfer principle*, a celebrated result in NSA that states that reals and hyperreals share “the same properties”;
- The first-order language  $\mathcal{L}_X$  that specifies formulas in which syntax, precisely, are preserved by the transfer principle; and finally
- The semantical construct of *superstructure* for interpreting  $\mathcal{L}_X$ -formulas.

What is of paramount importance is the transfer principle; in order to formulate it in a mathematically rigorous manner, the two last items (the language  $\mathcal{L}_X$  on the syntactic side, and superstructures on the semantical side) are used. The first-order language  $\mathcal{L}_X$  is essentially that of set theory and has two predicates  $=$  and  $\in$ . The *superstructure*  $V(X)$  is then a semantical “universe” for such formulas, constructed from the base set  $X$ : concretely  $V(X)$  is the union of  $X$ ,  $\mathcal{P}(X)$ ,  $\mathcal{P}(X \cup \mathcal{P}(X))$ , and so on. Finally, when we take  $X = \mathbb{R}$  then the set  ${}^*X = {}^*\mathbb{R}$  is that of *hyperreals*; and the transfer principle claims that  $A$  holds for reals if and only if  ${}^*A$ —a formula essentially the same as  $A$ —holds for hyperreals. Its precise statement is:

**Lemma 3.7** (the transfer principle). *For any closed formula  $A$  in  $\mathcal{L}_X$ , the following are equivalent.*

- *The formula  $A$  is valid in the superstructure  $V(X)$ .*
- *The  $*$ -transform  ${}^*A$  of  $A$ —this is a formula in the language  $\mathcal{L}_{{}^*X}$ —is valid in the superstructure  $V({}^*X)$ .*

The transfer principle guarantees that we can employ the same abstract interpretation framework, for reals and hyperreals alike—*literally* the same, in the sense that we express the framework in the language  $\mathcal{L}_{\mathbb{R}}$ . Concretely, various constructions and meta results (such as soundness and termination) in abstract interpretation will be expressed as  $\mathcal{L}_{\mathbb{R}}$ -formulas, and since they are valid in  $V(\mathbb{R})$ , they are valid in the “nonstandard universe”  $V({}^*\mathbb{R})$  too, by the transfer principle.

*Hyperreals* We fix an *index set*  $I = \mathbb{N}$ , and an *ultrafilter*  $\mathcal{F} \subseteq \mathcal{P}(I)$  that extends the cofinite filter  $\mathcal{F}_c := \{S \subseteq I \mid I \setminus S \text{ is finite}\}$ . Its properties to be noted: 1) for any  $S \subseteq I$ , exactly one of  $S$  and  $I \setminus S$  belongs to  $\mathcal{F}$ ; 2) if  $S$  is *cofinite* (i.e.  $I \setminus S$  is finite), then  $S$  belongs to  $\mathcal{F}$ .

**Definition 3.8** (hyperreal  $r \in {}^*\mathbb{R}$ ). We define the set  ${}^*\mathbb{R}$  of *hyperreal numbers* (or *hyperreals*) by  ${}^*\mathbb{R} := \mathbb{R}^I / \sim_{\mathcal{F}}$ . It is therefore the set of infinite sequences on  $\mathbb{R}$  modulo the following equivalence  $\sim_{\mathcal{F}}$ : we have  $(a_0, a_1, \dots) \sim_{\mathcal{F}} (a'_0, a'_1, \dots)$  if

$$\{i \in I \mid a_i = a'_i\} \in \mathcal{F}, \quad \text{for which we say “} d_i = d'_i \text{ for almost every } i.” \quad (2)$$

A *hypernatural*  $n \in {}^*\mathbb{N}$  is defined similarly.

It follows that: two sequences  $(a_i)_i$  and  $(a'_i)_i$  that coincide except for finitely many indices  $i$  represent the same hyperreal. The predicates besides  $=$  (such as  $<$ ) are defined in the same way. A notable consequence is the existence of infinite numbers in the set of hyperreals and hypernaturals:  $\omega := [(1, 2, 3, \dots)]$  is a positive infinite since it is larger than any positive real  $r = [(r, r, \dots)]$  ( $i > r$  for almost every  $i \in \mathbb{N}$ ). In addition, the set of hyperreals includes infinitesimal numbers: a hyperreal  $\omega^{-1} := [(1, \frac{1}{2}, \frac{1}{3}, \dots)]$  is positive ( $0 < \omega^{-1}$ ) but is smaller than any (standard) positive real  $r$ .

*Superstructure* A *superstructure* is a “universe,” constructed step by step from a certain base set  $X$  (whose typical examples are  $\mathbb{R}$  and  ${}^*\mathbb{R}$ ). We assume  $\mathbb{N} \subseteq X$ .

**Definition 3.9** (superstructure). A *superstructure*  $V(X)$  over  $X$  is defined by  $V(X) := \bigcup_{n \in \mathbb{N}} V_n(X)$ , where  $V_0(X) := X$  and  $V_{n+1}(X) := V_n(X) \cup \mathcal{P}(V_n(X))$ .

The superstructure  $V(X)$  might seem to be a closure of  $X$  only under powersets, but it accommodates many set-forming operations. For example, ordered pairs  $(a, b)$  and tuples  $(a_1, \dots, a_m)$  are defined in  $V(X)$  as is usually done in set theory, e.g.  $(a, b) := \{\{a\}, \{a, b\}\}$ . The function space  $a \rightarrow b$  is thought of as a collection of special binary relations (i.e.  $a \rightarrow b \subseteq \mathcal{P}(a \times b)$ ), hence is in  $V(X)$ .

*The First-Order Language  $\mathcal{L}_X$*  We use the following first-order language  $\mathcal{L}_X$ , defined for each choice of the base set  $X$  like  $\mathbb{R}$  and  ${}^*\mathbb{R}$ .

**Definition 3.10** (the language  $\mathcal{L}_X$ ). *Terms* in  $\mathcal{L}_X$  consist of: variables  $x, y, x_1, x_2, \dots$ ; and a constant  $a$  for each entity  $a \in V(X)$ .

*Formulas* in  $\mathcal{L}_X$  are constructed as follows.

- The predicate symbols are  $=$  and  $\in$ ; both are binary. The *atomic formulas* are of the form  $s = t$  or  $s \in t$  (where  $s$  and  $t$  are terms).
- We allow Boolean combinations of formulas. We use the symbols  $\wedge, \vee, \neg$  and  $\Rightarrow$ .
- Given a formula  $A$ , a variable  $x$  and a term  $s$ , the expressions  $\forall x \in s. A$  and  $\exists x \in s. A$  are formulas.

Note that quantifiers always come with a bound  $s$ . The language  $\mathcal{L}_X$  depends on the choice of  $X$  (it determines the set of constants). We shall also use the following syntax

sugars in  $\mathcal{L}_X$ , as is common in set theory and NSA.

$(s, t)$  pair                       $(s_1, \dots, s_m)$  tuple                       $s \times t$  direct product  
 $s \subseteq t$  inclusion, short for  $\forall x \in s. x \in t$   
 $s(t)$  function application; short for  $x$  such that  $(t, x) \in s$   
 $s \circ t$  function composition,  $(s \circ t)(x) = s(t(x))$   
 $s \leq t$  inequality in  $\mathbb{N}$ ; short for  $(s, t) \in \leq$  where  $\leq \subseteq \mathbb{N}^2$

**Definition 3.11** (semantics of  $\mathcal{L}_X$ ). We interpret  $\mathcal{L}_X$  in the superstructure  $V(X)$  in the obvious way. Let  $A$  be a closed formula; we say  $A$  is *valid* if  $A$  is true in  $V(X)$ .

*The \*-Transform and the Transfer Principle* As we mentioned the transfer principle says that a closed formula  $A$  in the language  $\mathcal{L}_X$  is valid in  $V(X)$  if and only if  $*A$  in  $\mathcal{L}_{*X}$  is valid in  $V(*X)$ . We shall describe how we syntactically transform  $A$  in  $\mathcal{L}_X$  into  $*A$  in  $\mathcal{L}_{*X}$ .

For that purpose, in particular in translating constants in  $\mathcal{L}_X$  (for entities in  $V(X)$ ) to  $\mathcal{L}_{*X}$ , we will need the following *semantical* translation. The so-called *ultrapower construction* yields a canonical map

$$*(\_): V(X) \longrightarrow V(*X), \quad a \longmapsto *a \quad (3)$$

that is called the *\*-transform*. It is a map from the universe  $V(X)$  of standard entities to  $V(*X)$  of nonstandard entities. The details of its construction are in Appendix A or in [26].

The above map  $*(\_): V(X) \rightarrow V(*X)$  becomes a *monomorphism*, a notion in NSA. Most notably it will satisfy the *transfer principle* (Lem. 3.13).

**Definition 3.12** (\*-transform of formulas). Let  $A$  be a formula in  $\mathcal{L}_X$ . The *\*-transform* of  $A$ , denoted by  $*A$ , is a formula in  $\mathcal{L}_{*X}$  obtained by replacing each constant  $a$  occurring in  $A$  with the constant  $*a$  that designates the element  $*a \in V(*X)$ .

**Lemma 3.13** (the transfer principle). *For any closed formula  $A$  in  $\mathcal{L}_X$ ,  $A$  is valid (in  $V(X)$ ) if and only if  $*A$  is valid (in  $V(*X)$ ).*  $\square$

We can prove, for instance, the following proposition using the transfer principle (the proof is in Appendix C). This proposition has a practical implication: our implementation relies on it in simplifying formulas including the infinitesimal constant  $\text{dt}$ .

**Proposition 3.14.** *Let  $A$  be an  $\mathcal{L}_{\mathbb{R}}$ -formula with a unique free variable  $x$ ; to emphasize it we write  $A(x)$  for  $A$ . Then the validity of the formula*

$$\exists r \in \mathbb{R}. (0 < r \wedge \forall x \in \mathbb{R}. (0 < x < r \Rightarrow A(x)))$$

*(in  $V(\mathbb{R})$ ) implies the validity of  $*A(\text{dt})$  in  $V(*\mathbb{R})$ .*  $\square$

### 3.4 The Modeling Language WHILE<sup>dt</sup>

WHILE<sup>dt</sup>, a modeling language for hybrid systems based on NSA, is introduced in [33]. It is an augmentation of a usual imperative language (such as IMP in [35]) with a constant  $\text{dt}$  that expresses an infinitesimal number.

**Definition 3.15.** Let  $\mathbf{Var}$  be the set of variables. The syntax of  $\mathbf{WHILE}^{\text{dt}}$  is as follows:

$$\begin{aligned} \mathbf{AExp} \ni a &::= x \mid r \mid a_1 \text{ aop } a_2 \mid \text{dt} \\ &\quad \text{where } x \in \mathbf{Var}, r \in \mathbb{R} \text{ and } \text{aop} \in \{+, -, \cdot, /\} \\ \mathbf{BExp} \ni b &::= \text{true} \mid \text{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\ \mathbf{Cmd} \ni c &::= \text{skip} \mid x := a \mid c_1; c_2 \mid \text{if } b \text{ then } c_1 \text{ else } c_2 \mid \text{while } b \text{ do } c. \end{aligned}$$

An expression  $a \in \mathbf{AExp}$  is an *arithmetic expression*,  $b \in \mathbf{BExp}$  is a *Boolean expression* and  $c \in \mathbf{Cmd}$  is a *command*.

As we explained in §1, the infinitesimal constant  $\text{dt}$  enables us to model not only discrete dynamics but also continuous dynamics without explicit ODEs. For example, the water-level monitor is modeled as a  $\mathbf{WHILE}^{\text{dt}}$  program shown in Fig. 2. As another example, the thermostat can be modeled as the program on the right. One can see that the continuous dynamics modeled in this example is

```
(*Thermostat*)
x := 22; p := 0;
while true do {
  if p = 0 then x := x - 3 * x * dt
  else x := x + 3 * (30 - x) * dt;
  if x >= 22 then p := 0
  else {if x <= 18 then p := 1
        else skip
      }
}
```

**Fig. 4.** Thermostat in  $\mathbf{WHILE}^{\text{dt}}$

beyond piecewise-linear. Even dynamics defined by nonlinear ODEs can be modeled in  $\mathbf{WHILE}^{\text{dt}}$  in the same manner. To go further to accommodate an arbitrary hybrid automaton we must properly deal with *nondeterminism*, a feature currently lacking in  $\mathbf{WHILE}^{\text{dt}}$ . Although we expect that to be not hard, precise comparison between  $\mathbf{WHILE}^{\text{dt}}$  and hybrid automata in expressivity is future work.

In the usual, standard abstract interpretation (without  $\text{dt}$ ), a command  $c$  is assigned its *collecting semantics*  $\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$  (see e.g. [10]). This is semantics by reachable sets of memory states, as the concrete semantics. Presence of  $\text{dt}$  in the syntax of  $\mathbf{WHILE}^{\text{dt}}$  calls for an infinitesimal number in the picture. The first thing to try would be to replace  $\mathbb{R}$  with  $^*\mathbb{R}$ , and let  $\mathbf{WHILE}^{\text{dt}}$  commands interpreted as functions of the type  $\mathcal{P}(\mathbf{Var} \rightarrow ^*\mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow ^*\mathbb{R})$ . This however is not suited for the purpose of interpreting recursion in presence of  $\text{dt}$ .<sup>7</sup> We rely instead on our theory of *hyperdomains* that is used in [34] and described in Appendix B; see the interpretation of while loops in Table 1. This calls for the interpretation of commands to be of the type  $^*(\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}))$ , a subset of  $^*\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow ^*\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$ . The last type will be used in the following definition.

**Definition 3.16.** *Collecting semantics* for  $\mathbf{WHILE}^{\text{dt}}$ , in Table 1, has the following types where  $\mathbb{B}$  is  $\{\text{tt}, \text{ff}\}$ :  $\llbracket a \rrbracket : ^*(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow ^*\mathbb{R}$  for  $a \in \mathbf{AExp}$ ;  $\llbracket b \rrbracket : ^*(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathbb{B}$  for  $b \in \mathbf{BExp}$ ; and  $\llbracket c \rrbracket : ^*\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow ^*\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$  for  $c \in \mathbf{Cmd}$ .

In [33] and in §1, the semantics of a while loop is defined using the idea of sectionwise execution, instead of as a least fixed point. This is not suited for employing abstract interpretation—the latter is after all for computing least fixed points. The collecting semantics in Def. 3.16 (Table 1) does use least fixed points; it is based on the

<sup>7</sup> If we interpret commands as functions  $\mathcal{P}(\mathbf{Var} \rightarrow ^*\mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow ^*\mathbb{R})$ , the interpretation  $\llbracket \text{while } x < 10 \text{ do } x := x + \text{dt} \rrbracket \{(x \mapsto 0)\}$  by a least fixed point will be  $\{x \mapsto r \mid \exists n \in \mathbb{N}. r = n * \text{dt}\}$ , not  $\{x \mapsto r \mid \exists n \in ^*\mathbb{N}. r = n * \text{dt} \wedge r \leq 10\}$  as we expect. The problem is that *internality*—an “well-behavedness” notion in NSA—is not preserved in such a modeling.

$\llbracket x \rrbracket \sigma := \sigma(x)$ for each $x \in \mathbf{Var}$	$\llbracket \mathbf{true} \rrbracket \sigma := \mathbf{tt}$
$\llbracket r \rrbracket \sigma := r$ for each $r \in \mathbb{R}$	$\llbracket \mathbf{false} \rrbracket \sigma := \mathbf{ff}$
$\llbracket a_1 \text{ aop } a_2 \rrbracket \sigma := \llbracket a_1 \rrbracket \text{ aop } \llbracket a_2 \rrbracket$	$\llbracket b_1 \wedge b_2 \rrbracket \sigma := \llbracket b_1 \rrbracket \wedge \llbracket b_2 \rrbracket$
$\llbracket \mathbf{dt} \rrbracket \sigma := (1, \frac{1}{2}, \frac{1}{3}, \dots)$	$\llbracket \neg b \rrbracket \sigma := \neg(\llbracket b \rrbracket \sigma)$
$\llbracket \mathbf{skip} \rrbracket \mathbf{S} := \mathbf{S}$	
$\llbracket x := a \rrbracket \mathbf{S} := \{ \sigma \llbracket a \rrbracket \sigma / x \mid \sigma \in \mathbf{S} \}$	
$\llbracket c_1; c_2 \rrbracket \mathbf{S} := \llbracket c_2 \rrbracket (\llbracket c_1 \rrbracket \mathbf{S})$	
$\llbracket \mathbf{if } b \text{ then } c_1 \text{ else } c_2 \rrbracket \mathbf{S} := \begin{cases} \llbracket c_1 \rrbracket \sigma \mid \sigma \in \mathbf{S}, \llbracket b \rrbracket \sigma = \mathbf{tt} \\ \cup \{ \llbracket c_2 \rrbracket \sigma \mid \sigma \in \mathbf{S}, \llbracket b \rrbracket \sigma = \mathbf{ff} \} \end{cases}$	
$\llbracket \mathbf{while } b \text{ do } c \rrbracket := \text{lfp}(*\Phi(\llbracket b \rrbracket)(\llbracket c \rrbracket))$	
where $\Phi : (\mathbf{St} \rightarrow \mathbb{B} \cup \{\perp\}) \rightarrow (\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})) \rightarrow$	
$\left( (\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})) \rightarrow (\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}) \rightarrow \mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})) \right)$	
is defined by $\Phi(f)(g) = \lambda \psi. \lambda S. S \cup \psi \{ (g(\sigma)) \mid \sigma \in S, f(\sigma) = \mathbf{tt} \} \cup \{ \sigma \mid \sigma \in S, f(\sigma) = \mathbf{ff} \}$ .	

**Table 1.** WHILE<sup>dt</sup> collecting semantics

alternative WHILE<sup>dt</sup> semantics introduced in [27] (it will also appear in the forthcoming full version of [22, 33]). The equivalence of the two semantics is established in [27].

In the rest of the paper we restrict the set of variables  $\mathbf{Var}$  to be finite. This assumption—a realistic one when we focus on the program to be analyzed—makes our NSA framework much simpler. Therefore  $\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$  and  $*\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$  are equal to  $\mathcal{P}(\mathbb{R}^n)$  and  $*\mathcal{P}(\mathbb{R}^n)$  for some  $n \in \mathbb{N}$  respectively; we prefer the latter notations in what follows.

## 4 Abstract Interpretation Augmented with Infinitesimals

In the current section are our main theoretical contributions—a metatheory of *nonstandard abstract interpretation* that justifies the workflow in §2.2.

(Standard) abstract interpretation infrastructure such as Prop. 3.1 and Prop. 3.3 is not applicable to WHILE<sup>dt</sup> programs, since  $*\mathcal{P}(\mathbb{R}^n)$  is not a cpo.<sup>8</sup> Thus, building on the theoretical foundations in the above, we now extend the abstract interpretation framework for the analysis of WHILE<sup>dt</sup> programs (and the hybrid systems modeled thereby). We introduce an *abstract hyperdomain* over  $*\mathbb{R}$  as the transfer of the (standard, over  $\mathbb{R}$ ) domain of convex polyhedra. We then interpret WHILE<sup>dt</sup> programs in them, and transfer the three widening operators mentioned in §3.1 to the nonstandard setting. We classify them into *uniform* ones—for which termination is guaranteed even in the nonstandard setting—and non-uniform ones. The main theorems are Thm. 4.3 and Thm. 4.9, for soundness (in place of Prop. 3.1) and termination (in place of Prop. 3.3) respectively.

### 4.1 The Domain of Convex Polyhedra over Hyperreals

We extend convex polyhedra to the current nonstandard setting.

<sup>8</sup> One can see that the ascending chain defined by  $X_n := \{k * \mathbf{dt} \mid 0 \leq k \leq n\}$  does not have the supremum in  $*\mathcal{P}(\mathbb{R}^n)$  since  $\{k * \mathbf{dt} \mid k \in \mathbb{N}\}$  is not *internal* (see Appendix A).

**Definition 4.1** (convex polyhedra over  ${}^*\mathbb{R}$ ). A *convex polyhedron* on  $({}^*\mathbb{R})^n$  is an intersection of finite number of affine half-spaces on  $({}^*\mathbb{R})^n$ , that is, the set of points  $\mathbf{x} \in ({}^*\mathbb{R})^n$  that satisfy a certain finite set of linear inequalities. The set of all convex polyhedra on  $({}^*\mathbb{R})^n$  is denoted by  $\mathbb{CP}_n^{\mathbb{R}}$ .

**Proposition 4.2.** *The set  $\mathbb{CP}_n^{\mathbb{R}}$  of all convex polyhedra over  $({}^*\mathbb{R})^n$  is a (proper) subset of  ${}^*\mathbb{CP}_n$ , the  $*$ -transform of the (standard) domain of convex polyhedra over  $\mathbb{R}^n$ .  $\square$*

What lies in the difference between the two sets  $\mathbb{CP}_n^{\mathbb{R}} \subsetneq {}^*\mathbb{CP}_n$  is, for example, a disk as a subset of  $\mathbb{R}^2$  (hence of  ${}^*\mathbb{R}^2$ ). In  ${}^*\mathbb{CP}_2$  one can use a constraint system whose number of linear constraints is a hypernatural number  $m \in {}^*\mathbb{N}$ ; using e.g.  $m = \omega = [(0, 1, 2, \dots)]$  allows us to approximate a disk with progressive precision.

In the following development of nonstandard abstract interpretation, we will use  ${}^*\mathbb{CP}_n$  as an abstract domain since it allows transfer of properties of  $\mathbb{CP}_n$ . We note, however, that our over-approximation of the interpretation  $\llbracket c \rrbracket$  of a loop-free  $\text{WHILE}^{\text{dt}}$  program  $c$  is always given in  $\mathbb{CP}_n^{\mathbb{R}}$ , i.e. with finitely many linear inequalities.

## 4.2 Theory of Nonstandard Abstract Interpretation

Our goal is to over-approximate the collecting semantics for  $\text{WHILE}^{\text{dt}}$  programs (Table 1) on convex polyhedra over  ${}^*\mathbb{R}$ . As we mentioned at the beginning of this section, however, abstract interpretation infrastructure cannot be applied since  ${}^*\mathcal{P}(\mathbb{R}^n)$  is not a cpo. Fortunately it turns out that we can rely on the  $*$ -transform (§3.3) of the theory in §3.1, where it suffices to impose the cpo structure only on  $\mathcal{P}(\mathbb{R})$  and the  $*$ -continuity—instead of the (standard) continuity—on the function  $\llbracket c \rrbracket$ . This theoretical framework of *nonstandard abstract interpretation*, which we shall describe here, is an extension of the *transferred domain theory* studied in [4, 34]. Part of the latter is found also in Appendix B.

**Theorem 4.3.** *Let  $(L, \sqsubseteq)$  be a cpo;  $F : {}^*L \rightarrow {}^*L$  be a  $*$ -continuous function; and  $\perp \in {}^*L$  be such that  $\perp \sqsubseteq F(\perp)$ . Let  $(\bar{L}, \bar{\sqsubseteq})$  be a preorder;  $\gamma : \bar{L} \rightarrow L$  be a function such that  $\bar{a} \bar{\sqsubseteq} \bar{b} \Rightarrow \gamma(\bar{a}) \sqsubseteq \gamma(\bar{b})$  for all  $\bar{a}, \bar{b} \in \bar{L}$ ; and  $\bar{F} : {}^*\bar{L} \rightarrow {}^*\bar{L}$  be a  $*$ -continuous function that is monotone with respect to  ${}^*\bar{\sqsubseteq}$  and satisfies  $F \circ {}^*\gamma \sqsubseteq {}^*\gamma \circ \bar{F}$ . Note that  $({}^*\bar{L}, {}^*\bar{\sqsubseteq})$  is also a preorder. Assume further that  $\bar{x} \in {}^*\bar{L}$  is a prefixed point of  $\bar{F}$  (i.e.  $\bar{F}(\bar{x}) \bar{\sqsubseteq} \bar{x}$ ) such that  $\perp \sqsubseteq {}^*\gamma(\bar{x})$ .*

*Then  $\bar{x}$  over-approximates  $\text{lfp}_{\perp} F$ , that is,  $\text{lfp}_{\perp} F \sqsubseteq {}^*\gamma(\bar{x})$ .  $\square$*

Our goal is over-approximation of the semantics of iteration of a loop-free  $\text{WHILE}^{\text{dt}}$  program  $c$ , relying on Thm. 4.3. Towards the goal, the next step is to find a suitable  $\bar{F} : {}^*\bar{L} \rightarrow {}^*\bar{L}$  that “stepwise approximates”  $F = \llbracket c \rrbracket$ , the collecting semantics of  $c$ . The next result implies that the  $*$ -transformation of  $\llbracket \_ \rrbracket_{\text{CP}}$  (defined in a usual manner in standard abstract interpretation, as mentioned in §3.1) can be used in such  $\bar{F}$ .

**Proposition 4.4.** *Let  $(L, \sqsubseteq), (\bar{L}, \bar{\sqsubseteq}), \gamma : \bar{L} \rightarrow L$  satisfy the hypotheses in Thm. 4.3. Assume that a continuous function  $F : L \rightarrow L$  is stepwise approximated by a monotone function  $\bar{F} : \bar{L} \rightarrow \bar{L}$ , that is,  $F \circ \gamma \sqsubseteq \gamma \circ \bar{F}$ . Then the  $*$ -continuous function  ${}^*F : {}^*L \rightarrow {}^*L$  is over-approximated by the monotone and internal function  ${}^*\bar{F} : {}^*\bar{L} \rightarrow {}^*\bar{L}$ , i.e.  ${}^*F \circ {}^*\gamma \sqsubseteq {}^*\gamma \circ {}^*\bar{F}$ .  $\square$*

We summarize what we observed so far on nonstandard abstract interpretation by instantiating the abstract domain to  $^*\mathbb{CP}_n$ . In the following  $\llbracket c \rrbracket$  is from Def. 3.16.

**Corollary 4.5** (soundness of nonstandard abstract interpretation on  $^*\mathbb{CP}_n$ ). *Let  $c$  be a loop-free  $\text{WHILE}^{\text{dt}}$  command; and let  $\perp \in ^*(\mathcal{P}(\mathbb{R}^n))$  and  $\bar{x} \in ^*\mathbb{CP}_n$  be such that  $(\llbracket c \rrbracket_{\mathbb{CP}})(\bar{x}) \sqsubseteq^* \bar{x}$  and  $\perp \sqsubseteq^* \gamma_{\mathbb{CP}_n}(\bar{x})$ . Then we have  $\text{lf}_{\perp} \llbracket c \rrbracket \sqsubseteq^* \gamma_{\mathbb{CP}_n}(\bar{x})$ .  $\square$*

### 4.3 Hyperwidening and Uniform Widening Operators

Towards our goal of using Thm. 4.3, the last remaining step is to find a prefixed point  $\bar{x}$ , i.e.  $\bar{F}(\bar{x}) \sqsubseteq^* \bar{x}$ . This is where widening operators are standardly used; see §3.1.

We can try  $*$ -transforming a (standard) notion—a strategy that we have used repeatedly in the current section. This yields the following result, that has a problem that is discussed shortly.

**Theorem 4.6.** *Let  $(L, \sqsubseteq)$  be a preorder and  $\nabla : L \times L \rightarrow L$  be a widening operator on  $L$ . Let  $F : ^*L \rightarrow ^*L$  be a monotone and internal function; and  $\perp \in ^*L$  be such that  $\perp \sqsubseteq^* F(\perp)$ . The iteration hyper-sequence  $\langle X_i \rangle_{i \in ^*\mathbb{N}}$ —indexed by hypernaturals  $i \in ^*\mathbb{N}$ —that is defined by*

$$X_0 = \perp, \quad X_{i+1} = \begin{cases} X_i & (\text{if } F(X_i) \sqsubseteq^* X_i) \\ X_i \nabla F(X_i) & (\text{otherwise}) \end{cases} \quad \text{for all } i \in ^*\mathbb{N}$$

*reaches its limit within some hypernatural number of steps and the limit  $\bigsqcup_{i \in \mathbb{N}} X_i$  is a prefixed point of  $F$  such that  $\perp \sqsubseteq^* \bigsqcup_{i \in \mathbb{N}} X_i$ .  $\square$*

The problem of Thm. 4.6 is that the *finite-step convergence* of iteration sequences for the original widening operator (described in Prop. 3.3) is now transferred to *hyper-finite-step convergence*. This is not desired. All the entities from NSA that we have used so far are constructs in denotational semantics—whose only role is to ensure soundness of verification methodologies<sup>9</sup> and on which we never actually operate—and therefore their infinite/infinitesimal nature has been not a problem. In contrast, computation of the iteration hypersequence  $\langle X_i \rangle_{i \in ^*\mathbb{N}}$  is what we actually compute to over-approximate program semantics; and therefore its termination guarantee within  $i \in ^*\mathbb{N}$  steps (Thm. 4.6) is of no use.

As a remedy we introduce a new notion of *uniformity* of the (standard) widening operators. It strengthens the original termination condition (Def. 3.2) by imposing a uniform bound  $i$  for stability of arbitrary chains  $\langle x_i \rangle \in L^{\mathbb{N}}$ . Logically the change means replacing  $\forall \exists$  by  $\exists \forall$ .

**Definition 4.7** (uniform widening). Let  $(L, \sqsubseteq)$  be a preorder. A function  $\nabla : L \times L \rightarrow L$  is said to be a *uniform widening operator* if the following two conditions hold.

- (Covering) For any  $x, y \in L$ ,  $x \sqsubseteq x \nabla y$  and  $y \sqsubseteq x \nabla y$ .
- (Uniform termination) Let  $x_0 \in L$ . There exists a *uniform bound*  $i \in \mathbb{N}$  such that: for any ascending chain  $\langle x_k \rangle \in L^{\mathbb{N}}$  starting from  $x_0$ , there exists  $j \leq i$  at which the chain  $\langle y_k \rangle \in L^{\mathbb{N}}$ , defined by  $y_0 = x_0$  and  $y_{k+1} = y_k \nabla x_{k+1}$  for all  $k \in \mathbb{N}$ , stabilizes (i.e.  $y_j = y_{j+1}$ ).

<sup>9</sup> Recall that  $\text{WHILE}^{\text{dt}}$  is a *modeling* language and we do not execute them.



It is straightforward that uniform termination implies termination.

We investigate uniformity of some of the commonly-known widening operators on convex polyhedra.

**Theorem 4.8.** *Among the three widening operators in §3.1,  $\nabla_S$  (Def. 3.5) and  $\nabla_M$  (Def. 3.6) are uniform, but  $\nabla_N$  ([3]) is not.*  $\square$

For example, the widening operator  $\nabla_S$  is uniform because once the first element  $x_0$  of an iteration sequence is fixed, the length of the iteration sequence is at most the number of linear inequalities that define the convex polyhedra  $x_0$ . However,  $\nabla_N$  is not uniform because an iteration sequence can be arbitrarily long even if the first element of it is fixed,

The following theorem is a “practical” improvement of Thm. 4.6; its proof relies on instantiating the uniform bound  $i$  in a suitable  $\mathcal{L}_{\mathbb{R}}$ -formula with a Skolem constant, before transfer.

**Theorem 4.9.** *Let  $(L, \sqsubseteq)$  be a preorder and  $\nabla \in L \times L \rightarrow L$  be a uniform widening operator on  $L$ . Let  $F : *L \rightarrow *L$  be a monotone and internal function; and  $\perp \in L$  be such that  $*\perp \sqsubseteq F(*\perp)$ . The iteration sequence  $\langle X_i \rangle_{i \in \mathbb{N}}$  defined by*

$$X_0 = *\perp, \quad X_{i+1} = \begin{cases} X_i & (\text{if } F(X_i) \sqsubseteq X_i) \\ X_i \nabla F(X_i) & (\text{otherwise}) \end{cases} \quad \text{for all } i \in \mathbb{N}$$

*reaches its limit within some finite number of steps; and the limit  $\bigsqcup_{i \in \mathbb{N}} X_i$  is a prefixed point of  $F$  such that  $*\perp \sqsubseteq \bigsqcup_{i \in \mathbb{N}} X_i$ .*  $\square$

Note that uniformity of  $\nabla$  is a *sufficient condition* for the termination of nonstandard iteration sequences (by  $*\nabla$ ); Thm. 4.9 does not prohibit other useful widening operators in the nonstandard setting. Furthermore, there can be a useful (nonstandard) widening operator except for the ones  $*\nabla$  that arise via standard ones  $\nabla$ .

It is a direct consequence of Thm. 4.9 and Thm. 4.8 that the analysis of  $\text{WHILE}^{\text{dt}}$  programs on  $*\mathbb{CP}_n$  is terminating with  $\nabla_S$  or  $\nabla_M$ .

## 5 Implementation and Experiments

### 5.1 Implementation

We implemented a prototype tool for analysis of  $\text{WHILE}^{\text{dt}}$  programs. The tool currently supports:  $*\mathbb{CP}_n$  as an abstract domain; and  $*\nabla_M$ ,  $*$ -transformation of  $\nabla_M$  in Def. 3.6 as a widening operator. Its input is a  $\text{WHILE}^{\text{dt}}$  program. It outputs a convex polyhedron that over-approximates the set of reachable memory states for each modes (or the values of discrete variables). Our tool consists principally of the following two components: 1) an OCaml frontend for parsing, forming an iteration sequence and making the set  $M$  for  $*\nabla_M$ ; and 2) a Mathematica backend for executing operations on convex polyhedra. The two components are interconnected by a C++ program, via MathLink.

There are some libraries such as Parma Polyhedra Library [2] that are commonly used to execute operations on convex polyhedra. They cannot be used in our implementation because we have to handle the infinitesimal constant  $\text{dt}$  as an truly infinitesimal

value. Instead we implemented Chernikova’s algorithm [6–8, 29] symbolically, using *computer algebra system (CAS)* on Mathematica based on Prop. 3.14.

Prop. 3.14 ensures that the transformation from  $*A(dt)$  to  $\exists r \in \mathbb{R}. (0 < r \wedge \forall x \in \mathbb{R}. (0 < x < r \Rightarrow A(x)))$  does not violate the soundness of the analysis. When we have to evaluate a formula including  $dt$ , we instead resolve  $\exists r \in \mathbb{R}. (0 < r \wedge \forall x \in \mathbb{R}. (0 < x < r \Rightarrow A(x)))$  using CAS (e.g. quantifier elimination).

## 5.2 Experiments

We analyzed two  $\text{WHILE}^{dt}$  programs—the water-level monitor (Fig. 2) and the thermostat (Fig. 4)—with our prototype. The experiments were on Apple MacBook Pro with 2.6 GHz Dual-core Intel Core i5 CPU and 8 GB memory and the execution times are the average of 10 runs.

**Water-Level Monitor** This is a piecewise-linear dynamics and a typical example used in hybrid automata literature. Our tool automates the analysis presented in §2; the execution time was 22.151 sec.

**Thermostat** The dynamics of this example is beyond piecewise-linear. The nonstandard abstract interpretation successfully analyzes this example without explicit piecewise-linear approximation. We believe this result witnesses a potential of our approach. We skip how it analyzes this example since the procedure is the same as the water-level monitor case. Our tool executes in 2.259 sec. and outputs an approximation from which we obtain an invariant  $18 - 54 * dt \leq x \leq 22 + 24 * dt$ .

## 6 Conclusions and Future Work

We presented an extended abstract interpretation framework in which hybrid systems are *exactly* modeled as programs with infinitesimals. The logical infrastructure by *non-standard analysis* (in particular the *transfer principle*) establishes its soundness. Termination is also ensured for *uniform* widening operators. Our prototype analyzer automates the extended abstract interpretation on the domain of convex polyhedra.

Regrettably our current implementation is premature and does not compare—in precision or scalability—with the state-of-art tools for hybrid system reachability such as SpaceEx [17] and Flow\* [5]. In fact the two examples in §5.2 are the only ones that we have so far succeeded to analyze. For other examples—especially nonlinear ones, to which our framework is applicable in principle—the analysis results are too imprecise to be useful. To improve there are some possible directions of future work to enhance the precision and scalability. Firstly, we could utilize trace partitioning [30], narrowing operators (the use of narrowing operators in the domain of convex polyhedra is indicated in [23, §3.4]) and other techniques that have been introduced to enhance the precision of the analysis. Secondly, we believe abstract domains such as *ellipsoids* [14], or some new ones that are tailored to nonlinear dynamics, can improve our analyzer. Finally, the lack of scalability is mainly due to our current way of eliminating  $dt$  (namely via Prop. 3.14): it relies on *quantifier elimination (QE)* that is highly expensive. A faster alternative is desired.

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## A Further on NSA in Superstructure

The definitions and results listed below are all well-established and commonly used in NSA. We follow [26, Chap. II], in which more details can be found.

**Remark A.1** (choice of the index set  $I$ ). In §3.3 we used the set  $\mathbb{N}$  of natural numbers as the index set  $I$ . It is common in NSA, however, to use  $I$  that is bigger than  $\mathbb{N}$ , and an ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(I)$  over  $I$ . The merit of doing so is that the resulting monomorphism  $*(\_)$  (see below) can be chosen to be an *enlargement*; see [26, Chap. II]. In what follows, however, we favor concreteness and keep using  $I = \mathbb{N}$  as the index set.

The transfer principle is a powerful result and we rely on it in the subsequent developments. Here are the first examples of its use; they are proved by transferring a suitable formula  $A$ .

**Lemma A.2.** 1. For  $a \in V(X) \setminus X$  we obtain an injective map

$$*(\_): a \longrightarrow *a, \quad (b \in a) \longmapsto (*b \in *a) \quad (4)$$

as a restriction of  $*(\_)$  in (3).

2. If  $a$  is a finite set, the map (4) is an isomorphism  $a \xrightarrow{\cong} *a$ .
3. Let  $a \rightarrow b$  be the set of functions from  $a$  to  $b$ . We have  $*(a \rightarrow b) \subseteq *a \rightarrow *b$ .
4.  $*(a_1 \times \cdots \times a_m) = *a_1 \times \cdots \times *a_m$ ; and  $*(a_1 \cup \cdots \cup a_m) = *a_1 \cup \cdots \cup *a_m$ .
5. For a binary relation  $r \subseteq a \times a$ , we have  $*r \subseteq *a \times *a$ . Moreover,  $r$  is an order if and only if  $*r$  is an order.  $\square$

*Internal Sets* The distinction between *internal* and *external* entities is central in NSA. In this paper however it is much of formality, since all the entities we use are internal. Here we present only the relevant definitions, leaving their intuitions to [26, §II.6]. In Appendix B, especially Rem. B.7, we will see that being internal is crucial for transfer.

**Definition A.3** (internal entity). An element  $b \in V(*X)$  is *internal* with respect to  $*(\_): V(X) \rightarrow V(*X)$  if there is  $a \in V(X)$  such that  $b \in *a$ . It is *external* if it is not internal.

**Lemma A.4.** A function  $f: *a \rightarrow *b$  is internal if and only if  $f \in *(a \rightarrow b)$ .  $\square$

*The Ultrapower Construction* We collect some necessary facts about the ultrapower construction of the monomorphism  $*(\_)$  in (3). Its details are beyond our scope; they are found in [26, §II.4].

The map  $*(\_)$  in fact factorizes into the following three steps.

$$\begin{array}{ccc} V(X) & \xrightarrow{*(\_)} & V(*X) \\ \downarrow \overline{(\_)} & & \uparrow M \\ \bigcup_{n \in \mathbb{N}} (V_n(X) \setminus V_{n-1}(X))^I & \xrightarrow{[\_]} & \prod_{\mathcal{F}}^0 V(X) \end{array} \quad (5)$$

The first factor  $\overline{(\_)}$  maps  $a \in V(X)$  to the constant function  $\bar{a}$  such that  $\bar{a}(i) = a$  for each  $i \in I$ ; recall that we have chosen  $I = \mathbb{N}$  (Rem. A.1). The second  $[\_]$  takes a

quotient modulo the ultrafilter  $\mathcal{F}$ ; finally the third factor  $M$  is the so-called *Mostowski collapse*.

For an intuition let us exhibit these maps in the simple setting of §3.3. The first factor  $\overline{(\_)}$  corresponds to forming constant streams:  $a \mapsto \bar{a} = (a, a, \dots)$ . The second  $[\_]$  is quotienting modulo  $\sim_{\mathcal{F}}$  of (2). The third map  $M$  does nothing—it is a book-keeping function that is only needed in the extended setting of superstructures.

The next result [26, Thm. 4.5] is about “starting from the lower-left corner” in (5). It follows from the definition of  $M$  and is a crucial step in the proof of the transfer principle (Lem. 3.13). It serves as an important lemma, too, later for the semantics of  $\text{WHILE}^{\text{dt}}$ .

**Lemma A.5** (Łoś’ theorem). *Let  $A$  be a formula in  $\mathcal{L}_X$  with its free variables contained in  $\{x_1, \dots, x_m\}$ ; and  $a_1, \dots, a_m \in \bigcup_{n \in \mathbb{N}} (V_n(X) \setminus V_{n-1}(X))^I$ . Then*

$$\begin{aligned} & {}^*A[M[a_1]/x_1, \dots, M[a_m]/x_m] \text{ is valid} \\ \iff & \{i \in I \mid A[a_1(i)/x_1, \dots, a_m(i)/x_m] \text{ is valid}\} \in \mathcal{F} . \end{aligned}$$

As a special case, let  $S \in V(X)$ , then

$$M[a] \in {}^*S \iff a(i) \in S \text{ for almost every } i. \quad \square$$

**Corollary A.6.** *Let  $a, b \in V(X)$ ; and for each  $i \in I$ ,  $f_i \in (a \rightarrow b)$  and  $x_i \in a$ . Then  $M[(f_i)_{i \in I}]$  is an internal function  ${}^*a \rightarrow {}^*b$ ; and  $M[(x_i)_{i \in I}] \in {}^*a$ . Moreover,*

$$M[(f_i(x_i))_{i \in I}] = \left( M[(f_i)_{i \in I}] \right) \left( M[(x_i)_{i \in I}] \right) . \quad \square$$

## B Appendix: Domain Theory, Transferred

The collecting semantics of  $\text{WHILE}^{\text{dt}}$  is introduced by solving recursive equations on  ${}^*\mathcal{P}(\mathbb{R}^n)$ . Here we present necessary theoretical foundations—they are like in [4, §2.2] and [34]—identifying the set  ${}^*\mathcal{P}(\mathbb{R}^n)$  as a hyperdomain and  $*$ -transferring domain theory.

The current section is an adaptation of what appeared in the appendix of [34]; and the definitions and results are similar to those in [4, §2.2], where what we call a hyperdomain is called an *internal domain*, and a  $*$ -continuous function is called an *internal continuous function*. The way we formulate these notions is however a bit different: we favor more explicit use of  $*$ -transforms, since this aids deductive verification via the transfer principle.

**Definition B.1.** In what follows we employ the theory of NSA presented in Appendix A. As the base set of a superstructure  $V(X)$  (Def. 3.9), we take  $X = \mathbb{R} \cup \mathbb{B} \cup \mathbf{Var}$ .

**Definition B.2** (hyperdomain). A *hyperdomain* is the pair of  $*$ -transforms  $({}^*D, {}^*\sqsubseteq)$  of a cpo  $(D, \sqsubseteq)$ .

**Example B.3.** The set  $\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})$  is a complete lattice with respect to the inclusion order  $\subseteq$ , therefore is a cpo. Its  $*$ -transfer  $(*(\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R})), * \subseteq)$  constitutes a hyperdomain.

We note that the set  $*(\mathcal{P}(\mathbf{Var} \rightarrow \mathbb{R}))$  coincides with the set of internal subsets of the space  $\{f : *\mathbf{Var} \rightarrow *\mathbb{R} \mid f \text{ is an internal function}\}$ . Moreover, under the assumption that  $\mathbf{Var}$  is a finite set (e.g. the set of variables occurring in a program  $c$ ), we can see that the last set  $\{f : *\mathbf{Var} \rightarrow *\mathbb{R} \mid f \text{ is an internal function}\}$  coincides with the function space  $\mathbf{Var} \rightarrow *\mathbb{R}$ . For this we use Lem. A.2.4.

Note that  $* \subseteq$  is an order in  $*D$  (Lem. A.2.5). Hyperdomain is the notion on which we wish to establish a suitable fixed point property.<sup>10</sup> Towards that goal, we first formulate the definitions of cpo and continuous function as  $\mathcal{L}_X$ -formulas, so that they can be transferred.

$$\begin{aligned} \text{BinRel}_{a,r} &::= r \subseteq a \times a & \text{Refl}_{a,r} &::= \forall x \in a. (x, x) \in r \\ \text{Trans}_{a,r} &::= \forall x, y, z \in a. ((x, y) \in r \wedge (y, z) \in r \Rightarrow (x, z) \in r) \\ \text{AntiSym}_{a,r} &::= \forall x, y \in a. ((x, y) \in r \wedge (y, x) \in r \Rightarrow x = y) \\ \text{Poset}_{a,r} &::= \text{BinRel}_{a,r} \wedge \text{Refl}_{a,r} \wedge \text{Trans}_{a,r} \wedge \text{AntiSym}_{a,r} \\ \text{HasBot}_{a,r} &::= \exists x \in a. \forall y \in a. (x, y) \in r \\ \text{AscCn}_{a,r}(s) &::= \forall x, x' \in \mathbb{N}. (x \leq x' \Rightarrow (s(x), s(x')) \in r) \\ \text{UpBd}_{a,r}(b, s) &::= \forall x \in \mathbb{N}. ((s(x), b) \in r) \\ \text{Sup}_{a,r}(p, s) &::= \text{UpBd}_{a,r}(p, s) \wedge \forall b \in a. (\text{UpBd}_{a,r}(b, s) \Rightarrow (p, b) \in r) \end{aligned}$$

Recall that the inclusion  $\mathbb{N} \subseteq X$  is assumed (Def. 3.9). These formulas are used in:

$$\begin{aligned} \text{CPO}_{a,r} &::= \text{Poset}_{a,r} \wedge \text{HasBot}_{a,r} \wedge \\ &\quad \forall s \in (\mathbb{N} \rightarrow a). (\text{AscCn}_{a,r}(s) \Rightarrow \exists p \in a. \text{Sup}_{a,r}(p, s)) , \\ \text{Conti}_{a_1, r_1, a_2, r_2}(f) &::= \forall s \in (\mathbb{N} \rightarrow a_1). \forall p \in a_1. \\ &\quad (\text{AscCn}_{a_1, r_1}(s) \wedge \text{Sup}_{a_1, r_1}(p, s) \Rightarrow \text{Sup}_{a_2, r_2}(f(p), f \circ s)) . \end{aligned} \tag{6}$$

**Definition B.4** ( $*$ -continuous function). Let  $(*D_1, * \subseteq_1)$  and  $(*D_2, * \subseteq_2)$  be hyperdomains. A function  $f : *D_1 \rightarrow *D_2$  is  $*$ -continuous if it is internal and satisfies the  $*$ -transform of the formula  $\text{Conti}_{D_1, \subseteq_1, D_2, \subseteq_2}$ . That is to be precise:  $*(\text{Conti}_{D_1, \subseteq_1, D_2, \subseteq_2})(f)$  is valid.<sup>11</sup> The set of  $*$ -continuous functions from  $*D_1$  to  $*D_2$  is denoted by  $*D_1 \rightarrow_{\text{ct}} *D_2$ .

**Lemma B.5.**  $(*D_1 \rightarrow_{\text{ct}} *D_2) = *(D_1 \rightarrow_{\text{ct}} D_2)$ . Here  $\rightarrow_{\text{ct}}$  denotes the set of continuous functions.

*Proof.* Assume  $f \in (*D_1 \rightarrow_{\text{ct}} *D_2)$ . The following closed formula is valid in  $V(X)$ :

$$\forall f' \in (D_1 \rightarrow D_2). (f' \in (D_1 \rightarrow_{\text{ct}} D_2) \Leftrightarrow \text{Conti}(f')) ,$$

<sup>10</sup> We believe an even more general setting is possible, by defining a hyperdomain to be an internal set  $D' \in V(*X)$  that satisfies a suitable formula like  $\text{CPO}_{a,r}$  in (6). Here we do not need such generality.

<sup>11</sup> We note that the condition is different from (somewhat informal) “ $*\text{Conti}_{D_1, * \subseteq_1, *D_2, * \subseteq_2}(f)$  is valid.” In the former a chain  $s$  ranges over internal functions  $s \in *(\mathbb{N} \rightarrow D_1)$ , while in the latter  $s$  can also be an external function  $\mathbb{N} \rightarrow *D_1$ .



where  $\text{Conti}$  is short for  $\text{Conti}_{D_1, \sqsubseteq_1, D_2, \sqsubseteq_2}$ . By transfer we have

$$\forall f' \in {}^*(D_1 \rightarrow D_2). (f' \in {}^*(D_1 \rightarrow_{\text{ct}} D_2) \Leftrightarrow {}^*\text{Conti}(f')) \quad (7)$$

valid in  $V({}^*X)$ . Thus  $f$  satisfies  ${}^*\text{Conti}(f')$ . Obviously  $f$  is internal; therefore  $f \in ({}^*D_1 \rightarrow_{\text{ct}} {}^*D_2)$ .

Conversely, assume  $f \in ({}^*D_1 \rightarrow_{\text{ct}} {}^*D_2)$ . By the definition of  ${}^*$ -continuity,  $f$  is internal, hence by Lem. A.4 we have  $f \in {}^*(D_1 \rightarrow D_2)$ . Moreover, using the definition of  ${}^*$ -continuity and (7), we have  $f \in {}^*(D_1 \rightarrow_{\text{ct}} D_2)$ .  $\square$

**Lemma B.6.** *Let  $({}^*D, {}^*\sqsubseteq)$  be a hyperdomain. Then a  ${}^*$ -continuous function  $f : {}^*D \rightarrow {}^*D$  has a least fixed point. Moreover, the function  ${}^*\mu$  that maps  $f$  to its least fixed point  $({}^*\mu)(f)$  is  ${}^*$ -continuous.*

*Proof.* By the usual construction in a cpo, we obtain the map

$$\mu : (D \rightarrow_{\text{ct}} D) \rightarrow_{\text{ct}} D, \quad f \mapsto \bigsqcup_{n \in \mathbb{N}} f^n(\perp).$$

Continuity of  $\mu$  is easy and standard. As its  ${}^*$ -transform we obtain a function  ${}^*\mu : ({}^*D \rightarrow_{\text{ct}} {}^*D) \rightarrow_{\text{ct}} {}^*D$ , where we used Lem. B.5 and A.2. The fact that  ${}^*\mu$  returns least fixed points is shown by the transfer of the following  $\mathcal{L}_X$ -formula.

$$\forall f \in (D \rightarrow_{\text{ct}} D). (f(\mu(f)) = \mu(f) \wedge \forall x \in D. (f(x) = x \Rightarrow \mu(f) \sqsubseteq x)) \quad \square$$

**Remark B.7.** It is crucial in the previous lemma that  $f : {}^*D \rightarrow {}^*D$  is an internal function. Specifically: recall that a formula  $A$  must be closed in order to be transferred (Lem. 3.13); and that in  $\mathcal{L}_X$  only bounded quantifiers ( $\forall x \in s$  with some bound  $s$ ) are allowed. For internal  $f$  we find such a bound by  $f \in {}^*(D \rightarrow D)$ ; for external  $f$  this is not possible.

## C Appendix: Omitted Proofs

### C.1 Proof of Thm. 3.14

*Proof.* Assume that

$$0 < r \wedge \forall x \in \mathbb{R}. (0 < x < r \Rightarrow A(x))$$

is valid for some  $r \in \mathbb{R}$ . By transfer,

$$0 < r \wedge \forall x \in {}^*\mathbb{R}. (0 < x < r \Rightarrow {}^*A(x))$$

is also valid for that  $r$ . This implies  ${}^*A(\text{dt})$  since  $0 < \text{dt} < r$  for any positive  $r \in \mathbb{R}$ .  $\square$

Hereafter in the proofs we use the following  $\mathcal{L}_{\mathbb{R}}$ -formulas.

**Definition C.1.** We define the following  $\mathcal{L}_{\mathbb{R}}$ -formulas:

$$\begin{aligned}
\text{Refl}_{L,\sqsubseteq} &:= \forall l \in L. (l, l) \in \sqsubseteq \\
\text{Trans}_{L,\sqsubseteq} &:= \forall l, m, n \in L. \left( ((l, m) \in \sqsubseteq \wedge (m, n) \in \sqsubseteq) \Rightarrow (l, n) \in \sqsubseteq \right) \\
\text{AntiSym}_{L,\sqsubseteq} &:= \forall l, m \in L. \left( ((l, m) \in \sqsubseteq \wedge (m, l) \in \sqsubseteq) \Rightarrow l = m \right) \\
\text{Preord}_{L,\sqsubseteq} &:= \text{Refl}_{L,\sqsubseteq} \wedge \text{Trans}_{L,\sqsubseteq} \\
\text{Poset}_{L,\sqsubseteq} &:= \text{Refl}_{L,\sqsubseteq} \wedge \text{Trans}_{L,\sqsubseteq} \wedge \text{AntiSym}_{L,\sqsubseteq} \\
\text{Concr}_{L_1,\sqsubseteq_1,L_2,\sqsubseteq_2,\gamma} &:= \forall \bar{x}, \bar{y} \in L_2. \bar{x} \sqsubseteq_2 \bar{y} \Rightarrow \gamma(\bar{x}) \sqsubseteq_1 \gamma(\bar{y}) \\
\text{AscCn}_{L,\sqsubseteq}(s) &:= \forall n, m \in \mathbb{N}. (n \leq m \Rightarrow s(n) \sqsubseteq s(m)) \\
\text{Sup}_{L,\sqsubseteq}(p, s) &:= (\forall n \in \mathbb{N}. s(n) \sqsubseteq p) \wedge \forall q \in L. \left( (\forall n \in \mathbb{N}. s(n) \sqsubseteq q) \Rightarrow p \sqsubseteq q \right) \\
\text{Cpo}_{L,\sqsubseteq} &:= \text{Poset}_{L,\sqsubseteq} \wedge \forall s \in \mathbb{N} \rightarrow L. (\text{AscCn}_{L,\sqsubseteq}(s) \Rightarrow \exists p \in L. \text{Sup}_{L,\sqsubseteq}(p, s)) \\
\text{Monotone}_{L_1,\sqsubseteq_1,L_2,\sqsubseteq_2}(f) &:= \forall x, y \in L_1. x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y) \\
\text{Conti}_{L_1,\sqsubseteq_1,L_2,\sqsubseteq_2}(f) &:= \forall s \in \mathbb{N} \rightarrow L_1. \forall p \in L_1. \\
&\quad \left( (\text{AscCn}_{L_1,\sqsubseteq_1}(s) \wedge \text{Sup}_{L_1,\sqsubseteq_1}(p, s)) \Rightarrow \text{Sup}_{L_2,\sqsubseteq_2}(f(p), f \circ s) \right) \\
\text{Basis}_{L,\sqsubseteq}(\perp, f) &:= \perp \sqsubseteq f(\perp) \\
\text{Cover}_{L,\sqsubseteq,\nabla} &:= \forall x, y \in L. (x \sqsubseteq x \nabla y) \wedge y \sqsubseteq x \nabla y \\
\text{Term}_{L,\sqsubseteq,\nabla} &:= \forall x \in \mathbb{N} \rightarrow L. \text{AscCn}(x) \Rightarrow \\
&\quad \left( \forall y \in \mathbb{N} \rightarrow L. \left( (y(0) = x(0) \wedge \forall n \in \mathbb{N}. y(n+1) = y(n) \nabla x(n+1)) \right. \right. \\
&\quad \left. \left. \Rightarrow \exists k \in \mathbb{N}. y(k) = y(k+1) \right) \right) \\
\text{Widen}_{L,\sqsubseteq,\nabla} &:= \text{Cover}_{L,\sqsubseteq,\nabla} \wedge \text{Term}_{L,\sqsubseteq,\nabla} \\
\text{WidenSeq}_{L,\sqsubseteq,\nabla}(X, \perp, F) &:= \\
&\quad X(0) = \perp \wedge \forall n \in \mathbb{N}. X(n+1) = X(n) \nabla F(X(n)).
\end{aligned}$$

## C.2 Proof of Prop. 4.2

*Proof.* The constraint system  $C$  for a (standard) convex polyhedron  $P \in \mathbb{CP}_n$  can be expressed by a pair  $(\mathbf{A}, \mathbf{b})$  of an  $m \times n$ -matrix  $\mathbf{A}$  and an  $m$ -vector  $\mathbf{b}$ , where  $m$  is the number of linear inequalities in  $C$ . The same applies to a (nonstandard) convex polyhedron  $P \in \mathbb{CP}_n^{\mathbb{R}}$ . For each of  $X \in \{\mathbb{R}, \mathbb{R}^*\}$ , let us denote, by  $\text{Constr}_{X,m,n}$ , the set of all convex polyhedra over  $X^n$  that can be expressed with  $m$  linear inequalities.

Then  $\mathbb{CP}_n = \bigcup_{m \in \mathbb{N}} \text{Constr}_{\mathbb{R},m,n}$  (with  $\bigcup_{m \in \mathbb{N}}$  expressed using an existential quantifier  $\exists m \in \mathbb{N}$ ) is a valid  $\mathcal{L}_{\mathbb{R}}$ -sentence by Def. 3.4. By the transfer principle (Lem. 3.13), we have a valid  $\mathcal{L}_{\mathbb{R}}$ -sentence  $*(\mathbb{CP}_n) = \bigcup_{m \in \mathbb{N}} \text{Constr}_{\mathbb{R}^*,m,n}$ . It has as its subset the set  $\mathbb{CP}_n^{\mathbb{R}} = \bigcup_{m \in \mathbb{N}} \text{Constr}_{\mathbb{R}^*,m,n}$  since  $\mathbb{N} \subseteq \mathbb{N}^*$ . This proves the claim.  $\square$

### C.3 Proof of Thm. 4.3

*Proof.* Let  $L, \bar{L} \in \mathbb{U}$  be sets,  $\sqsubseteq \in \mathcal{P}(L \times L)$  and  $\bar{\sqsubseteq} \in \mathcal{P}(\bar{L} \times \bar{L})$  be binary relations on  $L$  and  $\bar{L}$  respectively,  $\alpha : L \rightarrow \bar{L}$  and  $\gamma : \bar{L} \rightarrow L$  be functions. Then, the following  $\mathcal{L}_{\mathbb{R}}$ -sentence is valid (because it is equivalent to Prop. 3.1):

$$\begin{aligned} & \forall F \in L \rightarrow L. \forall \bar{F} \in \bar{L} \rightarrow \bar{L}. \forall \perp \in L. \forall \bar{x} \in \bar{L}. \\ & \left( \text{Cpo}_{L, \sqsubseteq} \wedge \text{Preord}_{\bar{L}, \bar{\sqsubseteq}} \wedge \text{Conti}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge \text{Monotone}_{\bar{L}, \bar{\sqsubseteq}, \bar{L}, \bar{\sqsubseteq}}(\bar{F}) \wedge \text{Concr}_{L, \sqsubseteq, \bar{L}, \bar{\sqsubseteq}, \gamma} \right. \\ & \wedge F \circ \gamma \sqsubseteq \gamma \circ \bar{F} \wedge \perp \sqsubseteq F(\perp) \wedge \perp \sqsubseteq \gamma(\bar{x}) \wedge \bar{F}(\bar{x}) \bar{\sqsubseteq} \bar{x} \\ & \left. \Rightarrow \text{lfp}_{\perp} F \sqsubseteq \gamma(\bar{x}) \right). \end{aligned}$$

By applying Lem. 3.13 to this  $\mathcal{L}_{\mathbb{R}}$ -sentence, we have the following valid  $\mathcal{L}_{\mathbb{R}}$ -sentence:

$$\begin{aligned} & \forall F \in *(L \rightarrow L). \forall \bar{F} \in *(\bar{L} \rightarrow \bar{L}). \forall \perp \in *L. \forall \bar{x} \in *\bar{L}. \\ & \left( *\text{Cpo}_{L, \sqsubseteq} \wedge *\text{Preord}_{\bar{L}, \bar{\sqsubseteq}} \wedge *\text{Conti}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge *\text{Monotone}_{\bar{L}, \bar{\sqsubseteq}, \bar{L}, \bar{\sqsubseteq}}(\bar{F}) \wedge *\text{Concr}_{L, \sqsubseteq, \bar{L}, \bar{\sqsubseteq}, \gamma} \right. \\ & \wedge F \circ *\gamma \sqsubseteq *\gamma \circ \bar{F} \wedge \perp \sqsubseteq *F(\perp) \wedge \perp \sqsubseteq *\gamma(\bar{x}) \wedge \bar{F}(\bar{x}) *\bar{\sqsubseteq} \bar{x} \\ & \left. \Rightarrow *\text{lfp}_{\perp} F \sqsubseteq *\gamma(\bar{x}) \right). \end{aligned}$$

This yields the statement of this theorem. For example, we can confirm that  $*\text{Concr}_{L, \sqsubseteq, \bar{L}, \bar{\sqsubseteq}, \gamma}$  holds from the following hypothesis in the theorem statement:  $\bar{a} \bar{\sqsubseteq} \bar{b} \Rightarrow \gamma(\bar{a}) \sqsubseteq \gamma(\bar{b})$  for all  $\bar{a}, \bar{b} \in \bar{L}$ . □

### C.4 Proof of Thm. 4.6

*Proof.* Let  $L \in \mathbb{U}$  be a set,  $\sqsubseteq \in \mathcal{P}(L \times L)$  be a binary relation on  $L$  and  $\nabla : L \times L \rightarrow L$  be a function. Then, the following  $\mathcal{L}_{\mathbb{R}}$ -sentence is valid (because it is equivalent to Prop. 3.3):

$$\begin{aligned} & \forall F \in L \rightarrow L. \forall \perp \in L. \forall X \in \mathbb{N} \rightarrow L. \\ & \text{Preord}_{L, \sqsubseteq} \wedge \text{Monotone}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge \text{Basis}_{L, \sqsubseteq}(\perp, F) \wedge \text{Widen}_{L, \sqsubseteq, \nabla} \\ & \wedge \text{WidenSeq}_{L, \sqsubseteq, \nabla}(X, \perp, F) \\ & \Rightarrow \exists i \in \mathbb{N}. \forall j \in \mathbb{N}. i \leq j \Rightarrow X(i) = X(j) \\ & \wedge \forall k \in \mathbb{N}. \left( (\forall l \in \mathbb{N}. k \leq l \Rightarrow X(k) = X(l)) \Rightarrow F(X(k)) \sqsubseteq X(k) \right). \end{aligned}$$

By applying Lem. 3.13 to this  $\mathcal{L}_{\mathbb{R}}$ -sentence, we have the following valid  $\mathcal{L}_{\mathbb{R}}$ -sentence:

$$\begin{aligned} & \forall F \in *(L \rightarrow L). \forall \perp \in *L. \forall X \in *(\mathbb{N} \rightarrow L). \\ & *\text{Preord}_{L, \sqsubseteq} \wedge *\text{Monotone}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge *\text{Basis}_{L, \sqsubseteq}(\perp, F) \wedge *\text{Widen}_{L, \sqsubseteq, \nabla} \\ & \wedge *\text{WidenSeq}_{L, \sqsubseteq, \nabla}(X, \perp, F) \\ & \Rightarrow \exists i \in *\mathbb{N}. \forall j \in *\mathbb{N}. i \leq j \Rightarrow X(i) = X(j) \\ & \wedge \forall k \in *\mathbb{N}. \left( (\forall l \in *\mathbb{N}. k \leq l \Rightarrow X(k) = X(l)) \Rightarrow F(X(k)) *\sqsubseteq X(k) \right) \end{aligned}$$

This yields the statement of this theorem. Note that the well-definedness of the iteration hyper-sequence (by induction on  $i \in {}^*\mathbb{N}$ ) is implicit in the above transfer arguments.  $\square$

### C.5 Proof of Thm. 4.9

*Proof.* We can characterize uniform widening operators as an  $\mathcal{L}_{\mathbb{R}}$ -sentence as follows (covering condition has been already expressed as an  $\mathcal{L}_{\mathbb{R}}$ -formula in Def. C.1):

$$\begin{aligned} \text{UnifTerm}_{L, \sqsubseteq, \nabla} &:= \forall x_0 \in L. \exists i \in \mathbb{N}. \forall x \in \mathbb{N} \rightarrow L. (\text{AscCn}(x) \wedge x(0) = x_0) \Rightarrow \\ &\left( \forall y \in \mathbb{N} \rightarrow L. \left( (y(0) = x(0) \wedge \forall n \in \mathbb{N}. y(n+1) = y(n) \nabla x(n+1)) \right. \right. \\ &\quad \left. \left. \Rightarrow \exists j \in \mathbb{N}. (j \leq i \wedge y(j) = y(j+1)) \right) \right) \\ \text{UnifWiden}_{L, \sqsubseteq, \nabla} &:= \text{Cover}_{L, \sqsubseteq, \nabla} \wedge \text{UnifTerm}_{L, \sqsubseteq, \nabla} \end{aligned}$$

Let  $L \in \mathbb{U}$  be a set,  $\sqsubseteq \in \mathcal{P}(L \times L)$  be a binary relation on  $L$  and  $\nabla : L \times L \rightarrow L$  be a function. Then, we can see directly that the following  $\mathcal{L}_{\mathbb{R}}$ -sentence is valid:

$$\forall \perp \in L. \exists i \in \mathbb{N}. \Psi(\perp)(i), \quad (8)$$

where

$$\begin{aligned} \Psi(\perp)(i) &= \\ &\forall F \in L \rightarrow L. \forall X \in \mathbb{N} \rightarrow L. \\ &\text{Preord}_{L, \sqsubseteq} \wedge \text{Monotone}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge \text{Basis}_{L, \sqsubseteq}(\perp, F) \wedge \text{UnifWiden}_{L, \sqsubseteq, \nabla} \\ &\wedge \text{WidenSeq}_{L, \sqsubseteq, \nabla}(X, \perp, F) \\ &\Rightarrow \forall j \in \mathbb{N}. i \leq j \Rightarrow X(i) = X(j) \\ &\wedge \forall k \in \mathbb{N}. \left( (\forall l \in \mathbb{N}. k \leq l \Rightarrow X(k) = X(l)) \Rightarrow F(X(k)) \sqsubseteq X(k) \right). \end{aligned}$$

Assume that  $\perp \in L$  is given. Then, by the  $\mathcal{L}_{\mathbb{R}}$ -sentence (8), there exists  $i \in \mathbb{N}$  such that  $\Psi(\perp)(i)$  holds. Therefore, by transferring  $\Psi(\perp)(i)$ ,  ${}^*\Psi(\perp)(i)$  holds for such  $i \in \mathbb{N}$ . Note that  ${}^*\Psi(\perp)(i)$  is the following  $\mathcal{L}_{\mathbb{R}}$ -sentence ( $\perp$  and  $i$  are dealt with as constants in the following  $\mathcal{L}_{\mathbb{R}}$ -sentence because  $\perp$  and  $i$  are defined outside the  $\mathcal{L}_{\mathbb{R}}$ -sentence):

$$\begin{aligned} &\forall F \in {}^*(L \rightarrow L). \forall X \in {}^*(\mathbb{N} \rightarrow L). \\ &{}^*\text{Preord}_{L, \sqsubseteq} \wedge {}^*\text{Monotone}_{L, \sqsubseteq, L, \sqsubseteq}(F) \wedge {}^*\text{Basis}_{L, \sqsubseteq}({}^*\perp, F) \wedge {}^*\text{UnifWiden}_{L, \sqsubseteq, \nabla} \\ &\wedge {}^*\text{WidenSeq}_{L, \sqsubseteq, \nabla}(X, {}^*\perp, F) \\ &\Rightarrow \forall j \in {}^*\mathbb{N}. i \leq j \Rightarrow X(i) = X(j) \\ &\wedge \forall k \in {}^*\mathbb{N}. \left( (\forall l \in {}^*\mathbb{N}. k \leq l \Rightarrow X(k) = X(l)) \Rightarrow F(X(k)) {}^*\sqsubseteq X(k) \right). \end{aligned}$$

This yields Thm. 4.9.  $\square$

### C.6 Proof of Thm. 4.8

We prove the uniformity and nonuniformity of three widening operators ( $\nabla_S$ ,  $\nabla_M$ ,  $\nabla_N$ ) in this order.

*Proof.* Let  $\langle X_i \rangle_i$  be a iteration sequence defined by  $\nabla_{\mathbb{CP}_n}$ , a basis  $X_0 = \text{con}(C_0)$  and a monotone function  $F$ . Let  $\langle C_i \rangle_i$  be the sequence of constraint systems that corresponds to  $\langle X_i \rangle_i$ . By definition of  $\nabla_{\mathbb{CP}_n}$  and the construction of  $\langle X_i \rangle_i$ , regardless of the function  $F$ ,  $C_{i+1} \subseteq C_i$  for all  $i \in \mathbb{N}$ . Thus for any basis  $X_0 = \text{con}(C_0)$  and monotone function  $F$ , we can reach a prefixed point by iterating the widening operator at most  $\#(C_0)$  times and this means the widening operator  $\nabla_{\mathbb{CP}_n}$  is uniform.  $\square$

*Proof.* The constraints in  $M$  may be added in the iteration sequence, but by the definition of the standard widening  $\nabla_S$ , a constraint in  $M$  will never appear once it is violated. Therefore the number of steps for an iteration sequence to converge is at most  $\#(M)$  larger than the case of standard widening.  $\square$

*Proof.* Assume that  $P_1 = \text{con}\{0 \leq x \leq 1, 0 \leq y \leq 1, z = 0\} \in \mathbb{CP}_3$ ,  $P_2 \in \mathbb{CP}_3$  includes  $P_1$  and the linear equation “ $z = 0$ ” is not included in  $C_2$ . Then  $P_1 \curvearrowright_N P_2$  holds because  $\#eq(C_1) > \#eq(C_2)$ . The maximum number of steps for an iteration sequence starting from  $P_2$  to converge is  $\#C_2$ . This is not limited uniformly because you can define  $P_2$  such that  $\#C_2$  is as large as you like.  $\square$

**Definition C.2** (Galois connection). Let  $(L, \sqsubseteq)$  and  $(\bar{L}, \bar{\sqsubseteq})$  be posets, and  $\alpha : L \rightarrow \bar{L}$  and  $\gamma : \bar{L} \rightarrow L$  be functions. A tuple  $((L, \sqsubseteq), (\bar{L}, \bar{\sqsubseteq}), \alpha, \gamma)$  is said to be a *Galois connection* if: for each  $x \in L$  and  $\bar{y} \in \bar{L}$ , we have  $\alpha x \bar{\sqsubseteq} \bar{y} \Leftrightarrow x \sqsubseteq \gamma \bar{y}$ . This fact is denoted by  $L \xrightleftharpoons[\gamma]{\alpha} \bar{L}$ ; and we call  $L$  a *concrete domain*,  $\bar{L}$  an *abstract domain*,  $\alpha$  an *abstraction function* and  $\gamma$  a *concretization function*.

**Proposition C.3.** A Galois connection  $(L, \sqsubseteq) \xrightleftharpoons[\gamma]{\alpha} (\bar{L}, \bar{\sqsubseteq})$  extends to monotone endofunctions. Concretely, it yields a Galois connection  $(L \xrightarrow[\text{mono.}]{\quad} L) \xrightleftharpoons[\tilde{\gamma}]{\tilde{\alpha}} (\bar{L} \xrightarrow[\text{mono.}]{\quad} \bar{L})$  where  $L \xrightarrow[\text{mono.}]{\quad} L$  and  $\bar{L} \xrightarrow[\text{mono.}]{\quad} \bar{L}$  are the posets of monotone functions ordered by the pointwise extension of  $\sqsubseteq$  and  $\bar{\sqsubseteq}$ . The functions  $\tilde{\gamma}$  and  $\tilde{\alpha}$  here are defined by:  $\tilde{\gamma}(f) = \gamma \circ f \circ \alpha$  and  $\tilde{\alpha}(f) = \alpha \circ f \circ \gamma$ , respectively.  $\square$

**Proposition C.4.** In the above setting, assume further that:  $\bar{F} : \bar{L} \rightarrow \bar{L}$  be a monotone function such that  $F \sqsubseteq \tilde{\gamma}(\bar{F})$ ; and that  $\bar{x} \in \bar{L}$  is a prefixed point of  $\bar{F}$  (i.e.  $\bar{F}(\bar{x}) \bar{\sqsubseteq} \bar{x}$ ) such that  $\alpha(\perp) \bar{\sqsubseteq} \bar{x}$ .

Then  $\bar{x}$  over-approximates  $\text{lfp}_{\perp} F$ , that is,  $\text{lfp}_{\perp} F \sqsubseteq \gamma(\bar{x})$ .  $\square$

**Definition C.5** (hyper-Galois connection). A *hyper-Galois connection*, denoted by  $*L \xrightleftharpoons[*\gamma]{*\alpha} *\bar{L}$ , is a quintuple  $(*L, *\bar{L}, *\alpha, *\gamma)$  of: the  $*$ -transform of a poset  $L$ ; that of a poset  $\bar{L}$ ; the  $*$ -transform  $*\alpha : *L \rightarrow *\bar{L}$  of a function  $\alpha : L \rightarrow \bar{L}$ ; and the  $*$ -transform  $*\gamma : *\bar{L} \rightarrow *L$  of  $\gamma$ . We require that the data  $(L, \bar{L}, \alpha, \gamma)$  forms a Galois connection (Def. C.2).

The above  $*\alpha$  is an internal function (i.e.  $*\alpha \in *(L \rightarrow \bar{L})$ ); see Appendix A for details. The notion of  $*$ -continuous function  $f' : *L \rightarrow *L$  is defined analogously, namely that  $f'$  is the  $*$ -transform of some continuous function  $f : L \rightarrow L$ . See Appendix B.

Here is the counterpart of Prop. C.4. As announced, it only requires the cpo structure of  $L$  (not of  $*L$ ) and the  $*$ -continuity of  $F$ .

**Theorem C.6.** *Let  $(L, \sqsubseteq)$  be a cpo,  $(\bar{L}, \bar{\sqsubseteq})$  be a poset such that  $L \xrightarrow[\gamma]{\alpha} \bar{L}$ , and consider the induced hyper-Galois connection  $*L \xrightarrow[\gamma]{\alpha} *\bar{L}$ . Let  $F : *L \rightarrow *L$  be a  $*$ -continuous function;  $\perp \in *L$  be such that  $\perp \sqsubseteq F(\perp)$ , and  $\bar{F} : *\bar{L} \rightarrow *\bar{L}$  be an internal function that is monotone with respect to  $\bar{\sqsubseteq}$ . Assume that  $F \sqsubseteq (*\gamma)(\bar{F})$ ; and that  $\bar{x} \in *\bar{L}$  is a prefixed point of  $\bar{F}$ , i.e.  $\bar{F}(\bar{x}) \bar{\sqsubseteq} \bar{x}$ .*

*Then  $\bar{x}$  over-approximates  $\text{lfp}_{\perp} F$ , that is,  $\text{lfp}_{\perp} F \sqsubseteq *\gamma(\bar{x})$ .* □